THE UNIVERSITY OF ALBERTA

CONTROL SYSTEM DESIGN BY EIGENVALUE ASSIGNMENT

by

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A THESIS

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ABSTRACT

This thesis presents the application of eigenvalue assignment techniques to both proportional and proportional plus integral control of multivariable linear systems using state feedback.

For proportional control system design, recursive design methods based on the eigenstructure of the open-loop system are considered in order to gain insight into the effects of available design freedom on the performance of the closed-loop system. The standard eigenvalue assignment techniques developed for proportion control systems are modified for the design of proportional plus integral controllers. A new theoretical result concerning a rank condition for the integral control matrix is presented as a formal proposition and proved.

Digital simulation studies of the eigenvalue assignment techniques applied to a double effect pilot plant evaporator model are described. The simulation results using proportional control schemes demonstrate that manipulation of eigenvalues only is not sufficient to ensure satisfactory performance of the closed-loop system; well-distributed closed-loop eigenvectors in the state space are also essential. The simulation results also reveal that available design options in recursive design methods have significant effects on both the closed-loop dynamics and the resulting feedback matrices.

Two modified eigenvalue assignment techniques for proportional plus integral control system design are applied to the

evaporator system. The simulation results of the evaporator system demonstrate that offsets in the selected state variables can be eliminated for any constant disturbance using proportional plus integral controllers designed by eigenvalue assignment techniques. However, the simulation results show that the transient behavior of the closed-loop system depends highly on the type of disturbance and the design method used.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER FOUR

Page

CHAPTER ONE

INTRODUCTION

The dynamic behavior of many chemical processes can be represented by a set of first-order ordinary differential equations which are linearized around normal operating conditions. The resulting state space representation of the system is in general a set of linear, time-invariant, multi-input, multi-output equations. It is well-known that the stability of such a system is exclusively determined by its eigenvalues and that the dynamic behavior of the system is governed by the modes (eigenvectors and eigenvalues) of the system [50].

In view of the important role of eigenvalues and eigenvectors in determining system dynamics, an obvious design approach is to synthesize a control system which ensures that the eigenvalues of the closed-loop system have specified values. For the class of proportional state feedback controllers, numerous design approaches for assigning closed-loop eigenvalues have been reported in the literature. Of these, two distinct approaches, one of which is based on the phase variable canonical representation of the system [1] and the other one based on the Jordan canonical representation (modal state representation) [41, 42] have received the most attention. Since steady state errors (offsets) due to unmeasured sustained disturbances are usually undesirable in process control, a natural extension of the eigenvalue assignment approach is the design of proportional plus integral controllers [7, 17, 37, 43]. The design of incomplete state feedback or output feedback control systems is currently under active research since not all

the states are accessible for measurement in many real processes [3, 6, 14]. However, the scope of this thesis is restricted to state feedback control systems and design approaches which are based on the modal representation. In addition to conceptual simplicity in design and implementation, this approach offers the significant advantage of giving more insight into system dynamics and providing more design freedom.

An important result of linear system theory [49] is that if a linear time-invariant system is state controllable, then state feedback control can be used to arbitrarily assign all of the eigenvalues of the closed-loop system. Despite the many design techniques available in the literature, reports of simulation or experimental studies of eigenvalue assignment techniques applied to process control problems are few in number. In this thesis, eigenvalue assignment techniques which employ state feedback are applied to a pilot plant double effect evaporator in digital simulation studies.

1.1 OBJECTIVES OF THE STUDY

The objectives of this study can be divided into two parts. The first part is intended to evaluate several eigenvalue assignment techniques for designing proportional controllers using state feedback. The methods of eigenvalue assignment considered in this study are variations of the approach of Simon [41, 42] and will be referred to as "Eigenvalue Assignment via Modal Analysis" throughout this thesis. This method is distinguished from other methods by having a feedback gain matrix of the form, $\underline{\underline{K}} = \sum_{i=1}^{p} \underline{g}_i \underline{f}_i^T$ where $\underline{g}_i$ and $\underline{f}_i$ are column vectors of appropriate dimensions. Superscript T denotes the transpose of a matrix and p is a positive integer which is

selected a priori by the designer. According to Simon's approach, for an arbitrary set of vectors $\{\underline{g}_i\}$ which preserves certain properties of the system, there exists a set of vectors $\{\underline{f}_i\}$ which enables the desired eigenvalue assignment. It is important to note that the feedback controller which assigns the desired eigenvalues to the closed-loop system is in general not unique.

It is the purpose of this first part to investigate the effects of the design freedom available in the choice of the set of arbitrary vectors $\{\underline{g}_i\}$ on the performance of the closed-loop system. More specifically, the following factors are evaluated via the simulation study:

- The effects of the closed-loop system matrix on the dynamic behavior of the system, e.g., if the eigenvalues of the closed-loop systems are the same, to what extent will the dynamic response of the closed-loop system change for different feedback control matrices?
- The effects of the desired closed-loop eigenvalues on the dynamic behavior of the system.
- The effects of the magnitude of feedback gains on the dynamic response of the system.
- The effects of the set of arbitrary vectors $\{\underline{g}_i\}$ on the dynamic response of the system and the resulting feedback gain matrix.
- Comparison of control laws derived by eigenvalue assignment and optimal control theory when both have the same closed-loop eigenvalues.

A further objective is to establish practical guidelines for choosing

the set of vectors $\{\underline{g}_i\}$ in such a way that design objectives such as low feedback gains, and satisfactory transient and steady state behavior of the system are satisfied.

In the second part of this study, existing eigenvalue assignment techniques are extended to the design of multivariable proportional plus integral controllers which eliminate steady state errors in some of the states. The condition that the integral controller matrix of the system must have full rank for arbitrary eigenvalue assignment is presented and proved in the form of a proposition. Based on this proposition, two practical algorithms are derived for designing proportional plus integral controllers. The performance of the resulting proportional plus integral feedback controllers is then compared with that of optimal proportional plus integral feedback controllers.

In both of these studies, the control laws are evaluated in simulation studies by applying them to a fifth-order state space model of pilot plant double effect evaporator.

1.2 STRUCTURE OF THE THESIS

This thesis consists of five chapters including this introductory chapter. In Chapter Two, a literature survey of eigenvalue assignment techniques for the design of proportional controllers is presented. Basic theorems and equations relevant to this study are also included in this chapter.

Chapter Three deals with the extension of eigenvalue assignment techniques to the design of proportional plus integral controllers. A necessary and sufficient condition for the realization

of proportional plus integral controllers which assign arbitrary eigenvalues to the system is presented in the form of a proposition. Two different algorithms based on this proposition are also included in this chapter together with their respective modifications to discrete-time systems.

Chapter Four describes the pilot plant double effect evaporator in the Department of Chemical Engineering and presents the results of simulation studies for proportional and proportional plus integral controllers.

Finally, Chapter Five summarizes the major results and conclusions of this thesis.

CHAPTER TWO

MULTIVARIBLE PROPORTIONAL CONTROLLER DESIGN

2.1 INTRODUCTION

Since Rosenbrock [40] first introduced modal control in 1962 as a possible design aid in the control of large chemical plants, considerable attention has been paid to control system design by assigning eigenvalues to the closed-loop system. Eigenvalue Assignment via Modal Analysis, mainly due to Simon [41, 42], is a powerful method in the sense that considerable design freedom exists and insight can be gained into the effect of moving eigenvalues on the system dynamics. The design freedom enables the feedback controller to be designed in either a single calculation (i.e., "the simultaneous approach") or in a recursive manner (i.e., "the recursive approach").

A brief survey of Eigenvalue Assignment via Modal Analysis is presented in Section 2.2 including both theory and applications to process control systems. Since eigenvalue assignability is closely related to the concept of state controllability, the development of the relationship is also included in this section. Other eigenvalue assignment techniques are briefly reviewed, for purposes of comparison.

In Section 2.3, basic equations and theory relevant to the development of this work are presented for both the simultaneous and recursive design approaches. However, major emphasis is put on the recursive design in view of the fact that more insight into the dynamic behavior of a system can be obtained by moving the eigenvalues one by one.

2.2 LITERATURE SURVEY

The conventional proportional feedback control of a single-loop first-order system can be interpreted as a special case of eigenvalue assignment since feedback control tends to decrease the time constant and hence shift the closed-loop eigenvalue. Actually, if a scalar is considered as an one-dimensional vector, this simple control scheme is an example of "ideal modal control" in the sense of Rosenbrock. However, eigenvalue assignment techniques are also attractive for the design of multivariable control systems since the stability of the system is guaranteed by assigning desired eigenvalues to the system.

Consider a time-invariant linear state space model and proportional feedback control system described by

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\ \underline{x}(t) + \underline{\underline{B}}\ \underline{u}(t) \qquad (\dot{\ } = d/dt) \tag{2-1}$$

$$\underline{u}(t) = \underline{\underline{K}}_{FB}\ \underline{x}(t) \tag{2-2}$$

where $\underline{x}(t)$ is an $n \times 1$ state vector, $\underline{u}(t)$ is an $m \times 1$ control vector and matrices $\underline{\underline{A}}$, $\underline{\underline{B}}$ and $\underline{\underline{K}}_{FB}$ are real, constant matrices with appropriate dimensions.

Ideal modal control was proposed by Rosenbrock [40] and may be defined as an eigenvalue assignment technique in which a certain number of eigenvalues of the open-loop system are changed while the eigenvectors and the other eigenvalues remain unaltered. To achieve ideal modal control, the control matrix $\underline{\underline{B}}$ must consist of the eigenvectors associated with eigenvalues to be changed [40]. However, in many applications, matrix $\underline{\underline{B}}$ is determined by the process model

and cannot be easily altered. Thus the degree to which ideal modal control can be achieved depends on how well matrix $\underline{\underline{B}}$ approximates the matrix of eigenvectors. Furthermore, there is no justification for preserving the open-loop eigenvectors unless these eigenvectors have favorable properties to begin with. However, systematic methods for synthesizing desired closed-loop eigenvectors are not available in the literature.

Ellis and White [8] modified Rosenbrock's modal control and introduced an eigenvalue assignment technique based on a modal analysis of the system. They abandoned the idea of keeping open-loop eigenvectors unchanged and were only concerned with shifting a selected eigenvalue to a desired location without affecting the other eigenvalues. However, their efforts were mainly concerned with shifting a single eigenvalue in single-input systems. In an analog computer simulation, they applied their technique to steam pressure control of an oil-fired boiler described by an eight-state equation and achieved better control in comparison with the best conventional three-term controller.

A fundamental question in eigenvalue assignment is under what conditions is it possible to assign an arbitrary set of eigenvalues to the system of Equation (2-1) by the control law of Equation (2-2). Considerable research has been carried out to answer this important question and a historical survey on this subject may be found in the paper by Willems and Mitter [46]. The fundamental and important result is that the eigenvalue assignability of a system is identical to the state controllability of that system.

This property is believed to have been known for a long time for single-input systems but for multi-input systems, a complete proof may be attributed to the independent work of Wonham [49] and Simon [41]. This important property may be cited in the form of a theorem [49].

> "The pair $(\underline{\underline{A}}, \underline{\underline{B}})$ is controllable if and only if, for every choice of the set Λ, there is a matrix $\underline{\underline{C}}$ such that $\underline{\underline{A}} + \underline{\underline{B}}\ \underline{\underline{C}}$ has Λ for its set of eigenvalues."

Although the final statement of the theorem is identical in Wonham's and Simon's work, they employed different approaches in the proof of the above theorem and these different approaches provide the basis of the two broad classes of eigenvalue assignment techniques that were mentioned in Chapter One.

2.2.1 Use of the Phase Variable Representation

Wonham's proof of the necessity part of the theorem was based on the linear transformation of the original system into the generalized phase variable canonical form. The transformed system matrix is in pseudo lower triangular form with blocks of phase variable canonical matrices (companion matrix forms) along the main diagonal. The generalized phase variable canonical form was used by Anderson and Luenberger [1] in their method of eigenvalue assignment. Wonham's major contribution can be interpreted as using the concept of cyclic subspaces to establish the existence of n linearly independent basis vectors which enable the system to be transformed into a generalized phase variable canonical form, if and only if the system is completely state controllable.

The phase variable canonical form has been widely used in shifting open-loop eigenvalues in single-input systems due to the fact that the characteristic polynomial of the system can be directly found from this form. A major part of the computation involves transforming the original system into the phase variable canonical form [18, 19]. Anderson and Luenberger's approach [1] may be viewed as a generalization of the single-input case to multi-input systems. However, in contrast to the single-input case, the representation of a multi-input system in the generalized phase variable canonical form is not unique, and consequently different control laws result depending on the canonical forms used. The non-uniqueness of the phase variable canonical form in multi-input systems provides design freedom for objectives such as regulation of the feedback gain elements [34] and assigning complex eigenvalues to the closed-loop system [36]. An extension of this method to systems which are partially controllable was also reported [44]. The advantage of this approach is the capability of handling complex and/or repeated eigenvalues but the approach suffers from the lack of information about eigenvectors. This information might be valuable in process control problems.

2.2.2 Use of the Jordan Canonical Form

In contrast with Wonham's approach, Simon's proof of the above theorem was based on a Jordan canonical (modal state) representation of the system and provides direct insight into eigenvalue assignability. His proof of the necessity part of the theorem is also constructive. It first makes sure that the system has distinct eigenvalues and then these eigenvalues are moved to the desired

locations. Assuming that the eigenvalues of matrix $\underline{\underline{A}}$ which are to be shifted are distinct, the postulated form of the feedback matrix is the sum of p dyadic products:

$$\underline{\underline{K}}_{FB} = \sum_{i=1}^{p} \underline{\underline{g}}_i \, \underline{f}_i^T \tag{2-3}$$

where p is the number of eigenvalues to be moved, $\underline{\underline{g}}_i$ is an $m \times 1$ vector, $\underline{f}_i$ is an $n \times 1$ vector. The structure of the gain matrix in Equation (2-3) provides a powerful tool in designing control system in such a way that eigenvalues are moved in a single step (the simultaneous approach), or recursively (the recursive approach) by specifying the sets of vectors $\{\underline{\underline{g}}_i\}$ and $\{\underline{f}_i\}$ independently.

Simultaneous and Recursive Approaches

In the simultaneous design method, each vector $\underline{\underline{g}}_i$ is assumed to be a constant multiple of a predefined vector $\underline{\underline{g}}_o$ in order to avoid having to solve a set of non-linear algebraic equations. This procedure has a net effect of converting a multi-input system into an equivalent single-input system. The vector $\underline{\underline{g}}_o$ may be chosen arbitrarily subject only to preserving the controllability of the original system. Although Power [35] suggested a quite lengthy and complicated method of choosing the vector $\underline{\underline{g}}_o$, no systematic way of selecting a satisfactory $\underline{\underline{g}}_o$ is available. Gould et al [11] extended this simultaneous design method to give an explicit gain formula for systems with repeated eigenvalues. Retallack and MacFarlane [39] presented an alternative approach which directly gives a compact form of the control law,

using the property of the return-difference determinant of a system. Other papers which adopted similar design approaches are also available [9, 24, 27, 29].

The form of the control law given in Equation (2-3) makes it possible to shift eigenvalues recursively in such a way that the p eigenvalues to be altered are divided into r ($\leq p$) groups, and each group of eigenvalues is moved to a desired location using the same number of $\underline{g}_i$'s and $\underline{f}_i$'s as the number of eigenvalues in that group. At each stage of the recursive design, the same procedure as was used in the simultaneous design is employed, and the intermediate closed-loop matrix and eigenvectors are calculated in order for use in the next stage. The design is completed by adding the individual control matrices, $\underline{g}_i\underline{f}_i^T$, that are calculated at each stage.

If only one eigenvalue is changed at a time, the eigenvectors of the intermediate closed-loop matrix can be updated from the previous set of eigenvectors by simple calculations. Ellis and White's method for eigenvalue assignment in single-input systems [8] may be regarded as a special case of this recursive design. Porter and Micklethwaite [27, 30] also reported a special case of this recursive design, derived independently from Simon, in which only one eigenvalue is moved to a desired location in each step and each eigenvalue is paired with a single control element at each step of the recursion. The pairing of eigenvalues and control variables can be achieved by specifying vector $\underline{g}_i$ to have only one non-zero element.

In addition to the number of recursive steps involved, the basic difference between the simultaneous and recursive design methods

stems from the fact that linearly independent vector $\underline{\underline{g}}_i$ are used in the recursive design while only one independent vector $\underline{\underline{g}}_o$ is used in the simultaneous design. Recursive design is a powerful approach due to its flexibility and the extra insight into the effect of feedback on the system dynamics which is gained by changing small numbers of eigenvalues at a time. Furthermore, the following corollaries provide the necessity for and the basis of, recursive design in handling a derogatory system, in which more than one independent eigenvector is associated with a distinct eigenvalue. By contrast, simultaneous design cannot change all of the repeated eigenvalues in this situation due to the unity rank of the resulting controller matrix.

> "Corollary 3.8 [42] If the input is a scalar quantity, i.e., $m = 1$, then a necessary condition for the system to be completely state controllable (completely state observable) is that no two Jordan blocks contain the same mode.
>
> Corollary 3.14 [42] If the pair $(\underline{\underline{A}}, \underline{\underline{B}})$ is completely state controllable and $\underline{\underline{K}}_{FB}$ is any $(m \times n)$ matrix, then the pair $(\underline{\underline{A}} + \underline{\underline{B}}\ \underline{\underline{K}}_{FB}, \underline{\underline{B}})$ is completely state controllable."

The approach of Simon and other workers for assigning eigenvalues, using the Jordan canonical form of a system, was termed "Eigenvalue Assignment via Modal Analysis" because the eigenvalues are shifted by analyzing the modal structure (eigen-

values and eigenvectors) of the open-loop system. Applications of modal analysis to other types of control problems are also found in the literature [5, 13, 23]. Levy et al [23] used modal analysis to provide insight into the dynamic response of a binary distillation column and to examine the validity of various column models. In their control system design, Davison and Chadha [5] and Howarth et al [13] selected the controls and states to be measured using modal information. Other than Ellis and White's application in the pressure control of a boiler, already mentioned, applications of eigenvalue assignment for the purpose of controlling a realistic system, using state feedback, are almost unavailable either in simulation or experimental studies. This is believed to be mainly due to the difficulty in getting all the state measurements. However, in the shift of a single eigenvalue using modal analysis, an interesting interpretation of the resulting gain-eigenvalue behavior of the closed-loop system obtained by a root-locus analysis was reported by Howarth et al [12].

For the class of incomplete state feedback control, where only very small numbers of controls and states are available, modifications of Rosenbrock's modal control were successfully applied to change a small number of eigenvalues [4, 5, 13].

2.3 THEORY

As in the case of most controller design methods, the theory of eigenvalue assignment techniques has been developed using a linear continuous-time model of a system. However, it is sometimes more convenient to represent a continuous system by a discretized model

based on a suitable sampling time, especially for purposes of digital-computer control. For this reason, the theory of Eigenvalue Assignment via Modal Analysis is presented for both continuous and discrete systems.

2.3.1 Continuous Systems

The dynamic behavior of many processes can be adequately approximated by the following linear time-invariant state space model,

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\ \underline{x}(t) + \underline{\underline{B}}\ \underline{u}(t) + \underline{\underline{D}}\ \underline{d}(t) \tag{2-4}$$

where $\underline{d}(t)$ is a $q \times 1$ disturbance vector, $\underline{\underline{D}}$ is a constant $n \times q$ matrix and the other symbols are defined below Equation (2-1).

The right eigenvector $\underline{w}_i$ and left eigenvector $\underline{v}_i$ associated with the i^{th} eigenvalue λ_i of matrix $\underline{\underline{A}}$ are defined by,

$$\underline{\underline{A}}\ \underline{w}_i = \lambda_i\ \underline{w}_i \tag{2-5}$$

$$\underline{v}_i^T\ \underline{\underline{A}} = \lambda_i\ \underline{v}_i^T\ . \tag{2-6}$$

Proper normalization of $\underline{w}_i$ and $\underline{v}_i$ gives the following property,

$$\begin{aligned} \langle \underline{w}_i, \underline{v}_j \rangle &= 1 \quad (\text{if } i = j) \\ &= 0 \quad (\text{if } i \neq j) \end{aligned} \tag{2-7}$$

where $\langle \cdot \rangle$ denotes the inner product of two vectors. Define matrices $\underline{\underline{W}}$, $\underline{\underline{V}}$ and $\underline{\underline{\Lambda}}$ such that $\underline{\underline{W}} = [\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_n]$, $\underline{\underline{V}} = [\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n]$ and $\underline{\underline{\Lambda}} = \text{diag}\ [\lambda_1, \lambda_2, \ldots, \lambda_n]$ assuming that the eigenvalues are distinct. Then, it can be easily shown that

$$\underline{\underline{\Lambda}} = \underline{\underline{V}}^T \underline{\underline{A}} \, \underline{\underline{W}} \quad . \tag{2-8}$$

To investigate the effects of feedback control on the eigenvalues of matrix $\underline{\underline{A}}$, a linear transformation, $\underline{x}(t) = \underline{\underline{W}} \, \underline{y}(t)$, is performed on the system of Equation (2-4). Then the resulting expression is given in terms of a new state vector $\underline{y}(t)$, (assuming for convenience, $\underline{d}(t) = \underline{0}$), by

$$\dot{\underline{y}}(t) = \underline{\underline{\Lambda}} \, \underline{y}(t) + \underline{\underline{H}} \, \underline{u}(t) \tag{2-9}$$

with

$$\underline{\underline{H}} = \underline{\underline{V}}^T \underline{\underline{B}} \quad . \tag{2-10}$$

From Equation (2-9), it is clear that the new state variables of the system are decoupled from each other and affected only by control vector $\underline{u}(t)$. Matrix $\underline{\underline{H}}$, which is frequently called "the mode controllability matrix" of the system, shows the effect of the controls on the eigenvalues of the system. It is important to observe that if all the elements of the i^{th} row of matrix $\underline{\underline{H}}$ are zero, the i^{th} eigenvalue of the system, λ_i, cannot be changed by any control scheme since the i^{th} mode (eigenvalue) is not controllable [41, 42].

In the absence of control, the dynamic response of the system of Equation (2-4) can be expressed in terms of eigenvectors and eigenvalues by [50],

$$\underline{x}(t) = \sum_{i=1}^{n} \langle \underline{v}_i, \underline{x}(0) \rangle e^{\lambda_i t} \underline{w}_i + \sum_{i=1}^{n} \int_o^t \langle \underline{v}_i, \underline{\underline{D}} \underline{d}(\tau) \rangle e^{\lambda_i (t-\tau)} d\tau \underline{w}_i \quad . \tag{2-11}$$

Equation (2-11) indicates that the stability of the system depends on

the eigenvalues, but the shape of transient response is also closely related to the orientation of the eigenvectors in the state space. Hence, if the control is mainly concerned with the regulation of some states, it is desirable to maintain control over both eigenvalues and eigenvectors in such a way that the combined effects of the eigenvalues and eigenvectors on these particular states are minimized.

After applying the control law given by Equation (2-2) to the system, the closed-loop system matrix $\underline{\underline{C}}$ that results, is

$$\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}}\ \underline{\underline{K}}_{FB} \quad . \tag{2-12}$$

The eigenvalues of matrix $\underline{\underline{C}}$ are determined by solving the characteristic equation $\det\,[\lambda \underline{\underline{I}}_n - \underline{\underline{C}}] = 0$, where $\underline{\underline{I}}_n$ denotes an identity matrix of dimension n. If the first p eigenvalues of matrix $\underline{\underline{A}}$ are assumed to be controllable and to be shifted to desired locations, ρ_i $(i = 1,2,\ldots,p)$, then the characteristic polynomial, $f(\lambda)$, of matrix $\underline{\underline{C}}$ can be written as

$$f(\lambda) = \det\,[\lambda \underline{\underline{I}}_n - \underline{\underline{C}}] \tag{2-13}$$

$$= \prod_{i=1}^{p} (\lambda-\rho_i) \prod_{j=1}^{n-p} (\lambda-\lambda_{p+j}) \quad .$$

However, from the assumed form of the feedback matrix given by Equation (2-3), the characteristic polynomial, $f(\lambda)$, becomes,

$$f(\lambda) = \det[\lambda \underline{\underline{I}}_n - \underline{\underline{A}} - \underline{\underline{B}} \sum_{i=1}^{p} \underline{g}_i \underline{f}_i^T] \quad . \tag{2-14}$$

In general, after choosing $\underline{f}_i$ to be equal to $\underline{v}_i$ and equating

Equations (2-13) and (2-14), the desired feedback matrix can be realized by solving the resulting non-linear equations for the $\underline{g}_i$'s [30, 41].

2.3.2 Simultaneous Design

In order to avoid solving non-linear algebraic equations, which results when one equates right hand sides of Equations (2-13) and (2-14) [41, 42], each vector $\underline{g}_i$ is assumed to be of the form of,

$$\underline{g}_i = \delta_i \underline{g}_o \quad (i = 1,2,\ldots,p) \tag{2-15}$$

where $\underline{g}_o$ is a specified constant vector of dimension m and δ_i is a constant multiplier to be determined. Then the characteristic polynomial, $f(\lambda)$, can be written as

$$f(\lambda) = \det[\lambda \underline{\underline{I}}_n - \underline{\underline{A}} - \underline{\underline{B}}\underline{g}_o \underline{\delta}^T \underline{\underline{V}}_p^T] \tag{2-16}$$

where $\underline{\delta}$ is a $p \times 1$ vector, $\underline{\underline{V}}_p$ is an $n \times p$ matrix and the i^{th} element of $\underline{\delta}$ and the i^{th} column of $\underline{\underline{V}}_p$ are denoted by δ_i and $\underline{v}_i$, respectively. Applying the Schur fomula [10] and using the relation given by Equation (2-8), the right hand side of Equation (2-16) can be factored into two determinants, hence

$$f(\lambda) = \det[\lambda \underline{\underline{I}}_n - \underline{\underline{\Lambda}}] \det[1 - \underline{\delta}^T (\lambda \underline{\underline{I}}_p - \underline{\underline{\Lambda}}_p) \underline{\alpha}] \tag{2-17}$$

where

$$\underline{\underline{\Lambda}}_p = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_p]$$

$$\underline{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_p]^T = \underline{\underline{V}}_p^T \underline{\underline{B}} \underline{g}_o \quad .$$

The second determinant in Equation (2-17) can be expanded to

$$\det[1-\underline{\delta}^T(\lambda \underline{\underline{I}}_p-\underline{\underline{\Lambda}}_p)^{-1}\underline{\alpha}] = 1 - \sum_{i=1}^{p} \frac{\delta_i \alpha_i}{(\lambda-\lambda_i)} \quad . \tag{2-18}$$

Suppose that δ_i are specified such that the following relation holds

$$1 - \sum_{i=1}^{p} \frac{\delta_i \alpha_i}{(\lambda-\lambda_i)} = \prod_{i=1}^{p} \frac{(\lambda-\rho_i)}{(\lambda-\lambda_i)} \quad . \tag{2-19}$$

Then from Equations (2-17) and (2-18), the characteristic polynomial $f(\lambda)$, becomes identical to Equation (2-13), thus assigning desired eigenvalues to the closed-loop system. If $\alpha_i \neq 0$, the constant coefficient δ_i can be determined from Equation (2-19) and is given by

$$\delta_i = - \frac{\sum_{j=1}^{p} (\rho_j-\lambda_i)}{\alpha_i \prod_{\substack{j=1 \\ j\neq i}}^{p} (\lambda_j-\lambda_i)} \qquad (i = 1,2,\ldots,p) \quad . \tag{2-20}$$

It is observed from Equation (2-20) that the underlying assumptions made in the derivation are that $\lambda_i \neq \rho_j$ $(i,j = 1,2,\ldots,p)$ and that $\underline{g}_o$ is chosen such that vector $\underline{\alpha}$ has non-zero elements.

2.3.3 Recursive Design

As mentioned in the previous section, the form of the feedback matrix given by Equation (2-3) suggests the possibility of shifting eigenvalues recursively. Although, an arbitrary number of eigenvalues can be shifted at each step of the recursion, only the case where one eigenvalue is moved to a desired location at each step, is dealt with here. From Equations (2-12) and (2-3), the closed-loop

matrix $\underline{\underline{C}}$ can be written as

$$\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}} \sum_{i=1}^{p} \underline{g}_i \underline{f}_i^T \quad . \tag{2-21}$$

Again, the first p eigenvalues of matrix $\underline{\underline{A}}$, λ_i $(i = 1,2,\ldots,p)$, are assumed to be shifted to new locations, ρ_i $(i = 1,2,\ldots,p)$, sequentially. The intermediate closed-loop matrix, $\underline{\underline{C}}_i$, after the i^{th} recursion, is defined by

$$\underline{\underline{C}}_i = \underline{\underline{C}}_{i-1} + \underline{\underline{B}}\underline{g}_i \underline{f}_i^T \qquad (i = 1,2,\ldots,p) \tag{2-22}$$

and

$$\underline{\underline{C}}_o = \underline{\underline{A}} \quad . \tag{2-23}$$

Clearly, Equation (2-22) shows that the closed-loop matrix $\underline{\underline{C}}_i$ after the i^{th} recursive step becomes the open-loop matrix for the $(i+1)^{th}$ step. Assuming that the i^{th} eigenvalue λ_i of matrix $\underline{\underline{A}}$ is moved to ρ_i in the i^{th} step, the desired objective can be achieved by prespecifying vector $\underline{g}_i$ arbitrarily and setting

$$\underline{f}_i = \delta_i \, \underline{v}_i^{(i-1)} \tag{2-24}$$

where δ_i is a constant to be determined and $\underline{v}_i^{(i-1)}$ is defined by the following relation,

$$\underline{\underline{C}}_{i-1}^T \, \underline{v}_i^{(i-1)} = \lambda_i \, \underline{v}_i^{(i-1)} \quad . \tag{2-25}$$

Equation (2-25) shows that $\underline{v}_i^{(i-1)}$ is the left eigenvector of $\underline{\underline{C}}_{i-1}$ associated with the i^{th} eigenvalue λ_i, which is to be moved at

the i^{th} step. By applying Equation (2-20), δ_i can be calculated:

$$\delta_i = \frac{(\rho_i - \lambda_i)}{\langle \underline{v}_i^{(i-1)}, \underline{\underline{B}}\underline{\underline{g}}_i \rangle} \quad (i = 1,2,\ldots,p) \quad . \tag{2-26}$$

Observe that each vector $\underline{\underline{g}}_i$ should be chosen such that $\langle\cdot\rangle$ in Equation (2-26) is not equal to zero and this is always possible since the first p eigenvalues of matrix $\underline{\underline{A}}$ are controllable. The rank of the resulting feedback matrix is less than or equal to m, depending on the number of independent vectors, $\underline{\underline{g}}_i$. By contrast, in the simultaneous design, the rank of the feedback control matrix is always unity. It is interesting to observe that the elements of vectors, $\underline{\underline{g}}_i$, can be interpreted as the ratio of control efforts employed in shifting the i^{th} eigenvalue [41]. If all the elements of vector $\underline{\underline{g}}_i$ are non-zero, all controls are used simultaneously and if only one element is specified to be non-zero, then a single control is used individually to shift each eigenvalue λ_i. Actually, Porter's sequential eigenvalue assignment algorithm [27, 30] is an example of the latter case.

2.3.4 Derivation of the Updated Right and Left Eigenvectors

In the recursive design method, it is necessary to calculate left eigenvectors at each recursive step, but if a small number of eigenvalues are shifted at each step, right and left eigenvectors of matrix $\underline{\underline{C}}_i$ can be easily updated from those of $\underline{\underline{C}}_{i-1}$ [41,42]. Again, only the case, where one eigenvalue is changed at a time, is described here. Although it is not absolutely necessary, the eigenvalues of matrix $\underline{\underline{C}}_{i-1}$ are assumed to be distinct for convenience, and vectors

$\underline{w}_i^{(i-1)}$ and $\underline{v}_i^{(i-1)}$ denote the right and left eigenvectors of $\underline{\underline{C}}_{i-1}$ associated with the i^{th} eigenvalue. Let the set of eigenvalues of matrix $\underline{\underline{C}}_i$, after the i^{th} recursive step, be $\{\lambda_j^{(i)}\}$ $(j = 1,2,\ldots,n)$; then $\lambda_j^{(i)} = \lambda_j^{(i-1)}$ (if $j \neq i$). From the definitions of an eigenvector and matrix $\underline{\underline{C}}_i$, it is easily verified that,

$$\begin{aligned} \underline{w}_j^{(i)} &= \underline{w}_j^{(i-1)} \qquad (\text{if } i \neq j) \\ &\neq \underline{w}_j^{(i-1)} \qquad (\text{if } i = j) \ . \end{aligned} \tag{2-27}$$

By the assumption of distinct eigenvalues of $\underline{\underline{C}}_{i-1}$, the set of vectors $\{\underline{w}_j^{(i-1)}\}$ $(j = 1,2,\ldots,n)$ spans the n-dimensional vector space, and hence it is possible to write,

$$\underline{w}_i^{(i)} = \sum_{j=1}^{n} q_j^{(i)} \underline{w}_j^{(i-1)} \ . \tag{2-28}$$

From the definition of an eigenvector and using the property given by Equation (2-7), it can be shown that [41]

$$\delta_i q_i^{(i)} \langle \underline{v}_j^{(i-1)}, \underline{\underline{B}}\underline{g}_i \rangle = q_j^{(i)} (\lambda_i^{(i)} - \lambda_j^{(i-1)}) \quad (j = 1,2,\ldots,n) \ . \tag{2-29}$$

However, considering the fact that $\lambda_j^{(i)} = \rho_j$ (if $j \leq i$) and $\lambda_j^{(i)} = \lambda_j$ (if $j > i$), $q_j^{(i)}$ can be determined from Equation (2-29) within a constant multiplier. The scale can be fixed by

$$\begin{aligned} q_j^{(i)} &= 1 \qquad (\text{if } j = i) \\ &= \frac{\delta_i \langle \underline{v}_j^{(i-1)}, \underline{\underline{B}}\underline{g}_i \rangle}{(\rho_i - \lambda_j^{(i-1)})} \qquad (\text{if } j \neq i) \ . \end{aligned} \tag{2-30}$$

The same analysis can be applied to update left eigenvectors, but by

using the property given by Equation (2-7), it can be readily verified that

$$\underline{v}_j^{(i)} = \underline{v}_j^{(i-1)} \qquad (\text{if} \quad j = i) \tag{2-31}$$

$$= \underline{v}_j^{(i-1)} - q_j^{(i)} \underline{v}_i^{(i-1)} \qquad (\text{if} \quad j \neq i) \quad .$$

2.3.5 Controller Gain Minimization

In many practical problems, it is desirable to put constraints on the magnitude of the controller gain elements because of limitations due to physical control variables, measurement noise, etc. In both simultaneous and recursive design methods, design freedom is provided by an arbitrary vector $\underline{g}_o$ or a set of arbitrary vectors $\{\underline{g}_i\}$ without affecting the eigenvalues of the final closed-loop matrix. If regulating the magnitude of the gain elements is a prime concern, these vectors can be chosen so as to minimize the magnitude of the largest gain element. In general, this will require the solution of a nonlinear optimization problem. However, if only one eigenvalue is shifted, Simon [41, 42] showed that the largest gain element can be minimized by selecting vector $\underline{g}_o$ to be

$$g_{o_j} = \text{sign} \langle \underline{v}_i, \underline{b}_j \rangle \qquad (j = 1,2,\ldots,m) \tag{2.32}$$

where g_{o_j} is the j^{th} element of vector $\underline{g}_o$, $\underline{v}_i$ is the left eigenvector, associated with the eigenvalue to be shifted and $\underline{b}_j$ is the j^{th} column of matrix $\underline{\underline{B}}$. Simon derived the relation of Equation (2-32) by defining "a measure of controllability" for the eigenvalue to be changed and proving that $\underline{g}_o$ chosen by Equation (2-32) maximizes this "measure of controllability".

Although Equation (2-32) is only valid when a single eigenvalue is shifted, this policy can be used in a recursive design to preserve the controllability of the system and to prevent excessively high gain magnitudes.

2.3.6 Discrete Systems

In many cases, the actual implementation of a control scheme can be more conveniently achieved on a discrete-time basis rather than on a continuous-time basis, especially in modern digital computer control systems. In such a case, the continuous model given by Equation (2-4) can be discretized into an equivalent discrete model [25],

$$\underline{x}[(k+1)T] = \underline{\underline{\phi}}(T)\underline{x}(kT) + \underline{\underline{\Delta}}(T)\underline{u}(kT) + \underline{\underline{\theta}}(T)\underline{d}(kT) \tag{2-33}$$

where

$$\underline{\underline{\phi}}(T) = e^{\underline{\underline{A}}T}$$

$$\underline{\underline{\Delta}}(T) = \left(\int_o^T e^{\underline{\underline{A}}\tau}\, d\tau\right)\underline{\underline{B}}$$

$$\underline{\underline{\theta}}(T) = \left(\int_o^T e^{\underline{\underline{A}}\tau}\, d\tau\right)\underline{\underline{D}}$$

and T is the sampling time of the system.

It can be easily shown that if the original continuous system is completely state controllable, then the discrete system given by Equation (2-33) is also completely state controllable [46]. Furthermore, the right and left eigenvectors are the same as the corresponding quantities of the continuous system and the i^{th} eigen-

value of the discrete system is $e^{\lambda_i T}$. If the control is based on discrete control law of the form of

$$\underline{u}(kT) = \underline{\underline{K}}_{FB}\ \underline{x}(kT) \tag{2-34}$$

then the design techniques described in previous subsections can be applied without any modification. However, to insure the stability of the resulting closed-loop system, the eigenvalues of this discrete system should be shifted inside of the unit circle in the complex plane.

CHAPTER THREE

MULTIVARIABLE PROPORTIONAL PLUS INTEGRAL CONTROLLER DESIGN

3.1 INTRODUCTION

The presence of sustained external disturbances is a common occurence in many practical control problems. It is well known that proportional control alone cannot eliminate undesirable steady-state errors (offsets) arising from sustained disturbances. Consequently, it is common practice to use integral feedback control in conjunction with proportional control for this purpose.

It is the objective of this chapter to consider the design of a class of multivariable proportional plus integral controllers (which will be referred to as PI controllers by applying eigenvalue assignment techniques. A survey of the relevant literature is presented in Section 3.2 with emphasis on the application of eigenvalue assignment techniques. The relationship between eigenvalue assignability and the state controllability is also reviewed in this section. In Section 3.3, a necessary and sufficient condition for realizing a PI control law which enables arbitrary assignment of all system eigenvalues is presented and proved. Two practical algorithms for assigning arbitrary closed-loop eigenvalues are also summarized in this section. In Section 3.3.2, these algorithms for continuous systems are modified for application to discrete systems.

3.2 LITERATURE SURVEY

For the class of state feedback control system, Johnson [15-17] was one of the first investigators who considered adding integral

control to proportional control to regulate states. He first considered the case of constant disturbances [15, 16] and excluded disturbance term from the state equation by adding an auxiliary state variable which represents the combined effect of control and disturbance terms. An essential assumption in this approach is that the range of the coefficient matrix, $\underline{\underline{D}}$, of the disturbance vector is contained in the range of the coefficient matrix, $\underline{\underline{B}}$, of the control vector. If the state equation satisfies this condition, he showed that offsets can be eliminated in n state variables by applying optimal control theory to the augmented system equation. In a later paper [17], Johnson extended his previous approach to the class of arbitrary vector disturbances, which satisfy a linear differential equation, and described controller design methods using both optimal control and stabilization (eigenvalue assignment) theory. Johnson's design approach using eigenvalue assignment technique guarantees the regulation of all state variables in the presence of a broader class of unmeasurable disturbances. However, in order for the essential assumption employed in his developments to be applicable to an arbitrary system, it is required that the number of controls be equal to the number of states. In many applications, this condition is difficult to meet because available controls are usually limited in number.

A different approach to this regulation problem using optimal control theory is given by Porter [25] and Newell and Fisher [21]. Their approach is different from Johnson's in that only as many integrated states, as preserve the controllability of the augmented system, are fed back, hence eliminating the range condition imposed

on the coefficient matrices of the control and disturbance vectors. However, this approach will eliminate offsets in only r states where $r \leq m$ and is valid for impulse or constant disturbances. Newell experimentally applied his approach to the double effect evaporator in the Department of Chemical Engineering and observed improvement over a proportional control scheme.

Before, proceeding further, it is convenient to define the problem mathematically. Consider the system described by Equation (2-4); if it is desired to eliminate offsets in r ($\leq n$) state variables, then a simple treatment is possible by defining a $r \times 1$ vector, $\underline{z}(t)$, by

$$\dot{\underline{z}}(t) = \underline{\underline{T}}_r \, \underline{x}(t) \tag{3-1}$$

where $\underline{\underline{T}}_r$ is a $r \times n$ matrix which consists of r appropriate rows of the $n \times n$ identity matrix $\underline{\underline{I}}_n$. Combining Equations (2-4) and (3-1) gives the augmented system:

$$\dot{\hat{\underline{x}}}(t) = \hat{\underline{\underline{A}}} \, \hat{\underline{x}}(t) + \hat{\underline{\underline{B}}} \, \hat{\underline{u}}(t) + \hat{\underline{\underline{D}}} \, \hat{\underline{d}}(t) \tag{3-2}$$

where

$$\hat{\underline{x}}(t) = \left[\begin{array}{c} \underline{x}(t) \\ \hline \underline{z}(t) \end{array} \right] \tag{3-3}$$

$$\hat{\underline{\underline{A}}} = \left[\begin{array}{c|c} \underline{\underline{A}} & \underline{\underline{0}} \\ \hline \underline{\underline{T}}_r & \underline{\underline{0}} \end{array} \right] \tag{3-4}$$

$$\hat{\underline{\underline{B}}} = \left[\begin{array}{c} \underline{\underline{B}} \\ \hline \underline{\underline{0}} \end{array} \right] \tag{3-5}$$

$$\hat{\underline{\underline{D}}} = \left[\begin{array}{c} \underline{\underline{D}} \\ \hline \underline{\underline{0}} \end{array} \right] \tag{3-6}$$

and the $\underline{\underline{0}}$'s are null matrices with appropriate dimensions.

The effect of augmenting the state vector is the introduction of r repeated zero eigenvalues to the system in Equation (3-2). These r zero eigenvalues should be moved to the left half of the complex plane in order to stabilize the system. The problem of regulation of r states is now converted to assigning at least r eigenvalues of the $(n+r) \times (n+r)$ system matrix, $\hat{\underline{\underline{A}}}$. In view of the relation between state controllability and eigenvalue assignability of a system [41, 49], it is necessary to examine the controllability of the augmented system. The conditions for the controllability of the pair $(\hat{\underline{\underline{A}}}, \hat{\underline{\underline{B}}})$ have been extensively investigated and necessary and sufficient conditions in terms of matrices $\underline{\underline{A}}$, $\underline{\underline{B}}$ and $\underline{\underline{T}}_r$ were reported by Porter and Power [31, 32, 38] and Davison and Smith [7, 43]. In their first paper [31], Porter and Power showed that r cannot exceed, m, the maximum rank of matrix $\underline{\underline{B}}$ and that a necessary condition for $(\hat{\underline{\underline{A}}}, \hat{\underline{\underline{B}}})$ to be a controllable pair is that the pair $(\underline{\underline{A}}, \underline{\underline{B}})$ must be controllable. They later prove that necessary and sufficient conditions for the controllability of the augmented system are that (i) the pair $(\underline{\underline{A}}, \underline{\underline{B}})$ is controllable and (ii) the rank of matrix $\underline{\underline{T}}_r(\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})^{-1}\underline{\underline{B}}$ is r, where matrix $\underline{\underline{K}}$ must be chosen to ensure the invertability of matrix $\underline{\underline{A}}$ [32, 38].

Davison and Smith have investigated the controllability of the augmented system with extension to the general case where matrix $\underline{\underline{T}}_r$ is arbitrary. They presented the same conditions given by Porter

and Power but in a more general form, and proved that if all the zero and positive eigenvalues of matrix $\hat{\underline{\underline{A}}}$ are moved to the left half of the complex plane, offsets in r states are eliminated for constant or impulse type disturbances. However, to cope with other types of disturbance such as a ramp or a parabolic disturbance etc., a proportional plus multiple-integral control system should be considered [7, 17]. The necessary and sufficient conditions for the controllability of the augmented system of Equation (3-2), which enables arbitrary eigenvalue assignment, are given by [7]:

i) $(\underline{\underline{A}}, \underline{\underline{B}})$ is a controllable pair and

ii) $$\text{rank} \left[\begin{array}{c|c} \underline{\underline{A}} & \underline{\underline{B}} \\ \hline \underline{\underline{T}}_r & \underline{\underline{0}} \end{array} \right] = n+r \quad .$$

Once the controllability of the augmented system of Equation (3-2) is ensured, then it is possible to shift an arbitrary number of eigenvalues of matrix $\hat{\underline{\underline{A}}}$ ($\leq n+r$) by designing a proportional controller, $\underline{u} = \hat{\underline{\underline{K}}}\, \hat{\underline{x}}$, for the system in Equation (3-2). However, if the simultaneous design approach described in Section 2.3.2 is applied to this augmented system, at most only one of the r repeated zero eigenvalues can be shifted due to the unity rank of the resulting feedback matrix [33]. This restriction is an obvious result of Corollary 3.8 cited in Section 2.2 and implies that **at most** only one state variable can be guaranteed to have zero steady state error. On the other hand, the recursive design approach described in Section 2.3.3 can be applied to shift an arbitrary number of eigenvalues of matrix $\hat{\underline{\underline{A}}}$ to the desired locations [29]. However, it should be kept in mind that a minimum of r steps of recursion are necessary to eliminate

offsets in r state variables and that the (n+r) dimensional eigenvectors of matrix $\hat{\underline{\underline{A}}}$ must be updated or recalculated at each recursive step even when shifting the eigenvalues of matrix $\underline{\underline{A}}$. Another possible approach is to apply Anderson and Luenberger's method of eigenvalue assignment to the system of Equation (3-2).

The special structures of matrix $\hat{\underline{\underline{A}}}$, $\hat{\underline{\underline{B}}}$ and $\hat{\underline{\underline{D}}}$ make it possible to decouple the task of assigning eigenvalues of the $(n+r) \times (n+r)$ system into two successive eigenvalue assignments for $(n \times n)$ and $(r \times r)$ systems. This approach was adopted by Power and Porter [37] in that, after first shifting the eigenvalues of the original system matrix $\underline{\underline{A}}$, the r zero eigenvalues of $\hat{\underline{\underline{A}}}$ are moved to desired locations simultaneously by inverting a suitably chosen $(r \times r)$ non-singular matrix. A similar design technique is derived in Section 3.3.1 from a condition concerning the rank of the integral feedback matrix.

Only a few applications of eigenvalue assignment approach to the design of PI controllers have been reported in the literature [4, 5]. Davison [4] applied an eigenvalue assignment technique to design the integral controller in the PI control of a boiler system described by 9 states and 2 controls, and a distillation column with 11 states and 3 controls. His simulation results showed that offsets were eliminated in those states subjected to integral action. Another application provided by Davison and Chadha [5] was concerned with the regulation of a single state variable of a large composite chemical plant of 41 states and 8 controls.

3.3 NEW RESULTS

The main new result is presented in Section 3.3.1 in the form of a proposition; two eigenvalue assignment algorithms are described based on this proposition. In Section 3.3.1, continuous-time systems are considered and the two algorithms are modified for application to discrete models in Section 3.3.2.

3.3.1 Continuous Systems

Consider the system described by Equation (3-2). The following proposition provides a statement of the restriction on the integral feedback matrix which guarantees the arbitrary assignment of all $n+r$ eigenvalues.

Proposition: Suppose that the system represented by Equation (3-2) is completely state controllable. Then a linear feedback control law of the form of Equation (3-7) which shifts all $n+r$ eigenvalues of matrix $\hat{\underline{\underline{A}}}$ to arbitrary non-zero values, which are either real numbers or complex conjugate pairs, can be realized if and only if the rank of the matrix, $\underline{\underline{K}}_I$, is equal to r.

$$\underline{u}(t) = [\underline{\underline{K}}_{FB} \mid \underline{\underline{K}}_I]\hat{\underline{x}}(t) \quad . \tag{3-7}$$

Proof: Combining Equations (3-7) and (3-2) give the closed-loop system matrix $\hat{\underline{\underline{C}}}$ as

$$\hat{\underline{\underline{C}}} = \left[\begin{array}{c|c} \underline{\underline{A}} + \underline{\underline{B}}\underline{\underline{K}}_{FB} & \underline{\underline{B}}\underline{\underline{K}}_I \\ \hline \underline{\underline{T}}_r & \underline{\underline{0}} \end{array}\right] \quad . \tag{3-8}$$

(Necessity) It will be shown that if the rank of matrix $\underline{\underline{K}}_I$ is less than r, the closed-loop system matrix $\hat{\underline{\underline{C}}}$ retains at least one zero eigenvalue even though the system of Equation (3-2) is completely state controllable. Matrix $\hat{\underline{\underline{C}}}$ can be expressed as the product of an $(n+r) \times (n+m)$ matrix and an $(n+m) \times (n+r)$ matrix as follows,

$$\hat{\underline{\underline{C}}} = \left[\begin{array}{c|c} \underline{\underline{A}} & \underline{\underline{B}} \\ \hline \underline{\underline{T}}_r & \underline{\underline{0}} \end{array}\right] \left[\begin{array}{c|c} \underline{\underline{I}}_n & \underline{\underline{0}} \\ \hline \underline{\underline{K}}_{FB} & \underline{\underline{K}}_I \end{array}\right] \tag{3-9}$$

Since the system of Equation (3-2) is completely state controllable, the first partitioned matrix in Equation (3-9) will have a rank of $(n+r)$. However, from the second partitioned matrix in Equation (3-9), it is clear that if matrix $\underline{\underline{K}}_I$ has a rank less than r, then the rank of matrix $\hat{\underline{\underline{C}}}$ becomes less than $(n+r)$ because the rank of the second matrix is less than $(n+r)$. This implies that at least one zero eigenvalue is retained. Thus the necessity part of the proposition is proved.

(Sufficiency) This part of the proof is constructive since a linear feedback control law of the form of Equation (3-7) is derived. Assume that $\underline{\underline{K}}_I$ has rank r. Denote the spectrum of eigenvalues of $\hat{\underline{\underline{A}}}$ by $\{\lambda_1,\lambda_2,\ldots,\lambda_n,0,\ldots,0\}$ and the spectrum of desired non-zero eigenvalues by $\{\rho_1,\rho_2,\ldots,\rho_n,\rho_{n+1},\ldots,\rho_{n+r}\}$.

It is convenient to write the control vector $\underline{u}(t)$ as

$$\underline{u}(t) = \underline{u}_1(t) + \underline{u}_2(t) \tag{3-10}$$

where $\underline{u}_1(t)$ will be used to shift the n eigenvalues of matrix $\underline{\underline{A}}$ and $\underline{u}_2(t)$ to shift the r zero eigenvalues of $\hat{\underline{\underline{A}}}$. Next, $\underline{u}_1(t)$ is

specified to be of the form

$$\underline{u}_1(t) = [\underline{\underline{K}} \mid \underline{\underline{0}}]\hat{\underline{x}}(t) \tag{3-11}$$

where $\underline{\underline{K}}$ is an $m \times n$ matrix. Combining Equations (3-10), (3-11) and (3-2) and assuming $\underline{d}(t) = \underline{0}$ for convenience, gives the following expression

$$\dot{\hat{\underline{x}}}(t) = \hat{\underline{\underline{A}}}_1\hat{\underline{x}}(t) + \hat{\underline{\underline{B}}}\underline{u}_2(t) \tag{3-12}$$

where

$$\hat{\underline{\underline{A}}}_1 = \left[\begin{array}{c|c} \underline{\underline{A}} + \underline{\underline{B}}\underline{\underline{K}} & \underline{\underline{0}} \\ \hline \underline{\underline{T}}_r & \underline{\underline{0}} \end{array}\right] . \tag{3-13}$$

The eigenvalues of the above system are the roots of the equation

$$\det [\lambda\underline{\underline{I}}_n - (\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})] \cdot \det [\lambda\underline{\underline{I}}_r] = 0 \quad . \tag{3-14}$$

Equation (3-14) indicates that only the n eigenvalues of matrix $\underline{\underline{A}}$ are affected by $\underline{u}_1(t)$. Since the pair $(\underline{\underline{A}},\underline{\underline{B}})$ is controllable, a proportional feedback matrix $\underline{\underline{K}}$ can be designed to assign the desired eigenvalue spectrum, $\{\rho_1,\rho_2,\ldots,\rho_n\}$, to matrix, $(\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})$.

After $\underline{\underline{K}}$ is specified, the modal matrix $\hat{\underline{\underline{V}}}$ of $\hat{\underline{\underline{A}}}_1^T$ has the form [32],

$$\hat{\underline{\underline{V}}}^T = \left[\begin{array}{c|c} \underline{\underline{V}}^T & \underline{\underline{0}} \\ \hline -\underline{\underline{T}}_r(\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})^{-1} & \underline{\underline{I}}_r \end{array}\right] \tag{3-15}$$

where $\underline{\underline{V}}$ is the modal matrix of $(\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})^T$. Performing the linear

transformation, $\hat{\underline{y}}(t) = \hat{\underline{\underline{V}}}^T \hat{\underline{x}}(t)$, Equation (3-12) is transformed into Jordan canonical form,

$$\dot{\hat{\underline{y}}}(t) = \left[\begin{array}{c|c} \underline{\underline{J}} & \underline{\underline{0}} \\ \hline \underline{\underline{0}} & \underline{\underline{0}} \end{array}\right] \hat{\underline{y}}(t) + \left[\begin{array}{c} \underline{\underline{H}}_1 \\ \hline \underline{\underline{H}}_2 \end{array}\right] \underline{u}_2(t) \tag{3-16}$$

where $\underline{\underline{J}}$ is the Jordan canonical form of $\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}}$ and

$$\underline{\underline{H}}_1 = \underline{\underline{V}}^T \underline{\underline{B}} \tag{3-17}$$

$$\underline{\underline{H}}_2 = -\underline{\underline{T}}_r (\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})^{-1} \underline{\underline{B}} \quad . \tag{3-18}$$

The control vector $\underline{u}_2(t)$ is specified to be

$$\underline{u}_2(t) = [\underline{\underline{0}} \mid \underline{\underline{K}}_I] \hat{\underline{y}}(t) \tag{3-19}$$

or, in terms of $\hat{\underline{x}}(t)$,

$$\underline{u}_2(t) = [-\underline{\underline{K}}_I \underline{\underline{T}}_r (\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})^{-1} \mid \underline{\underline{K}}_I] \hat{\underline{x}}(t) \quad . \tag{3-20}$$

Combining Equations (3-16) and (3-19) gives the closed-loop matrix in the modal domain, $\hat{\underline{\underline{C}}}_1$,

$$\hat{\underline{\underline{C}}}_1 = \left[\begin{array}{c|c} \underline{\underline{J}} & \underline{\underline{H}}_1 \underline{\underline{K}}_I \\ \hline \underline{\underline{0}} & \underline{\underline{H}}_2 \underline{\underline{K}}_I \end{array}\right] . \tag{3-21}$$

The $(n+r)$ eigenvalues of $\hat{\underline{\underline{C}}}_1$ are the roots of the characteristic equation,

$$\det[\lambda \underline{\underline{I}}_n - \underline{\underline{J}}] \cdot \det[\lambda \underline{\underline{I}}_r - \underline{\underline{H}}_2 \underline{\underline{K}}_I] = 0 \quad . \tag{3-22}$$

From Equation (3-22), it is evident that $\underline{\underline{K}}_I$ does not affect the

eigenvalues of $\underline{\underline{J}}$, which reduces the problem to assigning the eigenvalues of the $r \times r$ matrix, $\underline{\underline{H}}_2\underline{\underline{K}}_I$.

Matrix $\underline{\underline{K}}_I$ is specified to be of the form

$$\underline{\underline{K}}_I = \sum_{i=1}^{r} \underline{g}_i \underline{f}_i^T \tag{3-23}$$

where $\underline{g}_i$ is an arbitrary $m \times 1$ vector subject only to the controllability requirement and $\underline{f}_i$ is an $r \times 1$ vector. Since it is assumed that matrix $\underline{\underline{K}}_I$ has rank r, both $\{\underline{g}_i\}$ and $\{\underline{f}_i\}$ should be specified so that linearly independent vector sets result. A $r \times r$ matrix $\underline{\underline{C}}_i$ is defined recursively by

$$\underline{\underline{C}}_i = \underline{\underline{C}}_{i-1} + \underline{\underline{H}}_2 \underline{g}_i \underline{f}_i^T \qquad (i = 1,2,\ldots,r) \tag{3-24}$$

with

$$\underline{\underline{C}}_o = \underline{\underline{0}} \ . \tag{3-25}$$

Then, the recursive design method described in Section 2.3.3 can be directly applied to shift the r zero eigenvalues of $\underline{\underline{C}}_o$ by specifying

$$\underline{f}_i = \delta_i \underline{v}_i^{(i-1)} \tag{3-26}$$

where $\underline{v}_j^{(i)}$ is defined as the j^{th} left eigenvector associated with the j^{th} eigenvalue of $\underline{\underline{C}}_i$, however, from the Equation (3-25), $\underline{v}_j^{(o)}$ can be chosen arbitrarily but the set $\{\underline{v}_j^{(o)}\}$ should be linearly independent.

By combining Equations (3-10), (3-11) and (3-20), the resulting control law can be expressed as

$$\underline{u}(t) = [\underline{\underline{K}}-\underline{\underline{K}}_I\underline{\underline{T}}_r(\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})^{-1} \mid \underline{\underline{K}}_I]\hat{\underline{x}}(t) \quad . \tag{3-27}$$

This proposition serves as the basis for two algorithms for assigning eigenvalues. In both algorithms, the first step is to shift selected eigenvalues of $\underline{\underline{A}}$ to desired locations using proportional feedback control. The two proposed algorithms then differ in the design of the integral feedback matrix, $\underline{\underline{K}}_I$, which is used to shift the r zero eigenvalues.

Recursive Design: This algorithm closely follows the proof of the sufficiency part of the proposition in that the r zero eigenvalues are shifted to desired locations recursively. The algorithm consists of the following three steps.

Step 1) Design the $m \times n$ proportional control matrix $\underline{\underline{K}}$ so that matrix, $(\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})$, has the desired eigenvalues. This can be achieved using the algorithms mentioned in Chapter Two.

Step 2) The integral feedback matrix, $\underline{\underline{K}}_I$, is specified to be of the form

$$\underline{\underline{K}}_I = \sum_{i=1}^{r} \delta_i \underline{g}_i \underline{v}_i^{(i-1)^T} \quad . \tag{3-28}$$

Choose $\underline{g}_i$ so that $\langle \underline{v}_i^{(i-1)}, \underline{\underline{H}}_2\underline{g}_i \rangle \neq 0$ and the set, $\{\underline{g}_i\}$, is linearly independent. Since the augmented system of Equation (3-2) is completely state controllable, it is always possible to choose vectors $\underline{g}_i$ which satisfy these conditions. As shown in Section 2.3.3, δ_i can be calculated from the equation

$$\delta_i = \frac{\rho_{n+i}}{\langle \underline{v}_i^{(i-1)}, \underline{\underline{H}}_2\underline{g}_i \rangle} \quad (i = 1,2,\ldots,r) \quad . \tag{3-29}$$

A systematic way of specifying $\underline{\underline{g}}_i$ is to follow the procedure described in Section 2.3.5 which ensures the controllability of the system.

Step 3) Complete the design by constructing the feedback control law given by Equation (3-27).

This algorithm retains the computational advantage of the unity rank (dyadic) approach but yields a $\underline{\underline{K}}_I$ matrix of rank r, which shifts the r zero eigenvalues. The left eigenvectors, $\underline{v}_i^{(i-1)}$, which are required at each stage, can be updated by the method described in Section 2.3.4.

Simultaneous Design: Steps 1) and 3) of this algorithm are identical to those of Recursive Design. Consequently, only Step 2) is presented.

If the control law of Equation (3-27) is used, it follows from Equation (3-22) that the $(n+r)$ eigenvalues of the closed-loop system consist of the n roots of $\det[\lambda\underline{\underline{I}}_n-(\underline{\underline{A}}+\underline{\underline{B}}\underline{\underline{K}})] = 0$ and the r roots of $\det[\lambda\underline{\underline{I}}_r-\underline{\underline{H}}_2\underline{\underline{K}}_I] = 0$. This last equation can be used to design $\underline{\underline{K}}_I$.

Step 2) Let

$$\underline{\underline{H}}_2\underline{\underline{K}}_I = \underline{\underline{D}} \tag{3-30}$$

where $\underline{\underline{D}}$ is a $r \times r$ matrix whose eigenvalues are the r desired eigenvalues, ρ_{n+i} $(i = 1,2,\ldots,r)$. Furthermore, assume that $\underline{\underline{K}}_I$ has the form

$$\underline{\underline{K}}_I = \underline{\underline{H}}_2^T\underline{\underline{G}} \tag{3-31}$$

where $\underline{\underline{G}}$ is a $r \times r$ non-singular matrix. From Equations (3-30) and

(3-31), it follows that

$$\underline{\underline{H}}_2\underline{\underline{H}}_2^T\underline{\underline{G}} = \underline{\underline{D}} \; . \tag{3-32}$$

Since the $r \times m$ matrix $(r \leq m)$, $\underline{\underline{H}}_2$, has rank r, then the $(r \times r)$ matrix $\underline{\underline{H}}_2\underline{\underline{H}}_2^T$ also has rank r. Hence, Equation (3-32) can be rearranged to give

$$\underline{\underline{G}} = (\underline{\underline{H}}_2\underline{\underline{H}}_2^T)^{-1}\underline{\underline{D}} \; . \tag{3-33}$$

Combining Equations (3-31) and (3-33), matrix $\underline{\underline{K}}_I$ is expressed by

$$\underline{\underline{K}}_I = \underline{\underline{H}}_2^T(\underline{\underline{H}}_2\underline{\underline{H}}_2^T)^{-1}\underline{\underline{D}} \; . \tag{3-34}$$

The apparent difference between two algorithms is that the Recursive Design requires the calculation of eigenvectors, while the Simultaneous Design employs matrix inversion. This distinction is characteristic of all eigenvalue assignment techniques in the literature in that either eigenvector calculation or matrix inversion is required. However, the choice between the above two algorithms should be based on other factors such as magnitude of controller gains or performances of resulting systems rather than upon the computational difficulties involved.

Comparison With Existing Methods

As mentioned in the last part of Section 3.2, if the augmented system of Equation (3-2) satisfies the controllability conditions, eigenvalue assignment techniques developed for proportional control systems, such as Anderson and Luenberger's approach [1] and recursive design based on modal analysis [26, 41, 42], can be success-

fully applied to assign all eigenvalues of the augmented system. This explains why investigators in the field of eigenvalue assignment have been mainly concerned with establishing controllability conditions for the augmented systems. However, the disadvantage in using those existing techniques is that one has to deal with $(n+r)$ dimensional systems rather than n dimensional systems. Although r is usually much less than n, the computational burden increases drastically as the number of states increases. Actually, if as many controls as states are available, then probably Johnson's approach [17] is the most useful approach since it can cope with a general class of disturbances and the dimension of the augmented system is always $(n+1)$ if constant disturbances are assumed.

The two design algorithms derived in this section are based on the strategy of decoupling the eigenvalue assignment problem for the $(n+r) \times (n+r)$ system into two smaller problems, namely, assigning the eigenvalues of an $n \times n$ system and of a $r \times r$ system. This strategy is not new and has been used in the design of PI controllers by Johnson [17] and Power and Porter [37]. Its chief advantage is that only eigenvectors of the smaller subsystems need be calculated rather than the eigenvectors of the $(n+r) \times (n+r)$ system.

The second method given in this section is similar to that of Power and Porter [37] except a pseudo-inverse of matrix $\underline{\underline{H}}_2$ is used in the former, while the inverse of a matrix consisted of r linearly independent columns of matrix $\underline{\underline{H}}_2$ is used in the latter, for the design of the $\underline{\underline{K}}_I$ matrix. If m is equal to r, these two approaches will give identical results.

In the first method described in this section, although the basic design scheme is the same as used in Power and Porter's approach [37], r stages of recursion are used in designing the integral controller based on Corollary 3.8 given by Simon [41, 42].

3.3.2 Discrete Systems

If a discrete PI control law is preferred over a continuous one, the discrete controller may be derived by applying the eigenvalue assignment techniques to a discrete equivalent of Equation (3-2). Different discrete representations can be derived by several approaches. For example, one can directly discretize Equation (3-2), following standard procedures in the literature [25]; this gives the following equations:

$$\hat{\underline{x}}[(k+1)T] = \hat{\underline{\underline{\phi}}}(T)\hat{\underline{x}}(kT)+\hat{\underline{\underline{\Delta}}}(T)\underline{u}(kT)+\hat{\underline{\underline{\theta}}}(T)\underline{d}(kT) \tag{3-35}$$

where

$$\hat{\underline{x}}(kT) = \left[\begin{array}{c} \underline{x}(kT) \\ \hline \underline{z}(kT) \end{array}\right] . \tag{3-36}$$

Matrices $\hat{\underline{\underline{\phi}}}(T)$, $\hat{\underline{\underline{\theta}}}(T)$ and $\hat{\Delta}(T)$ are defined in the same way as matrices $\underline{\underline{\phi}}(T)$, $\underline{\underline{\theta}}(T)$ and $\underline{\underline{\Delta}}(T)$ are defined in Section 2.3.6. However, if $\underline{\underline{A}}$ is non-singular, matrices $\hat{\underline{\underline{\phi}}}(T)$, $\hat{\underline{\underline{\theta}}}(T)$ and $\hat{\underline{\underline{\Delta}}}(T)$ can be expressed compactly by †

$$\hat{\underline{\underline{\phi}}}(T) = \left[\begin{array}{c|c} \underline{\underline{\phi}}(T) & \underline{\underline{0}} \\ \hline \underline{\underline{T}}_r\underline{\underline{A}}^{-1}[\underline{\underline{\phi}}(T)-\underline{\underline{I}}_n] & \underline{\underline{I}}_r \end{array}\right] \tag{3-37}$$

†The details of the derivations of Equations (3.37)-(3-39) are presented in the Appendix.

$$\hat{\underline{\underline{\theta}}}(T) = \begin{bmatrix} \underline{\underline{\theta}}(T) \\ \hline \underline{\underline{T}}_r \underline{\underline{A}}^{-1}[\underline{\underline{\theta}}(T) - T\underline{\underline{D}}] \end{bmatrix} \tag{3-38}$$

$$\hat{\underline{\underline{\Delta}}}(T) = \begin{bmatrix} \underline{\underline{\Delta}}(T) \\ \hline \underline{\underline{T}}_r \underline{\underline{A}}^{-1}[\underline{\underline{\Delta}}(T) - T\underline{\underline{B}}] \end{bmatrix} . \tag{3-39}$$

As an alternative strategy, the discrete representation of the augmented system can be derived by combining Equation (2-33) with the following equation which defines a $r \times 1$ vector, $\underline{z}(kT)$, by

$$\underline{z}(kT) = T \sum_{i=o}^{k} \underline{\underline{T}}_r \underline{x}(iT) \qquad (k = 1,2,\ldots) \tag{3-40}$$

or equivalently

$$\underline{z}[(k+1)T] = T\, \underline{\underline{T}}_r \underline{x}[(k+1)T] + \underline{z}(kT) \quad . \tag{3-41}$$

Combining Equations (2-33) and (3.41) gives the augmented discrete version of the system, which has the same form of Equation (3-35). However, matrices $\hat{\underline{\underline{\phi}}}(T)$, $\hat{\underline{\underline{\theta}}}(T)$ and $\hat{\underline{\underline{\Delta}}}(T)$ are now modified to

$$\hat{\underline{\underline{\phi}}}(T) = \left[\begin{array}{c|c} \underline{\underline{\phi}}(T) & \underline{\underline{0}} \\ \hline T\underline{\underline{T}}_r \underline{\underline{\phi}}(T) & \underline{\underline{I}}_r \end{array}\right] \tag{3-42}$$

$$\hat{\underline{\underline{\theta}}}(T) = \begin{bmatrix} \underline{\underline{\theta}}(T) \\ \hline T\underline{\underline{T}}_r \underline{\underline{\theta}}(T) \end{bmatrix} \tag{3-43}$$

$$\hat{\underline{\underline{\Delta}}}(T) = \begin{bmatrix} \underline{\underline{\Delta}}(T) \\ \hline T\underline{\underline{T}}_r \underline{\underline{\Delta}}(T) \end{bmatrix} . \tag{3-44}$$

Newell and Fisher [21] used Equations (3-42) to (3-44) in their optimal PI controller design and experimentally proved the utility of these equations by applying the PI controllers to a pilot plant double effect evaporator.

Another discrete representation of the augmented system can be found in Porter and Crossley [29]. Their alternative definition of $\underline{z}(kT)$ is given by

$$\underline{z}(kT) = T \sum_{i=o}^{k-1} \underline{\underline{T}}_r \underline{x}(iT) \tag{3-45}$$

or equivalently

$$\underline{z}[(k+1)T] = T\underline{\underline{T}}_r \underline{x}(kT)+\underline{z}(kT) \quad . \tag{3-46}$$

Obviously, matrices $\hat{\underline{\underline{\phi}}}(T)$, $\hat{\underline{\underline{\theta}}}(T)$ and $\hat{\underline{\underline{\Delta}}}(T)$ are modified and now given by

$$\hat{\underline{\underline{\phi}}}(T) = \left[\begin{array}{c|c} \underline{\underline{\phi}}(T) & \underline{\underline{0}} \\ \hline T\underline{\underline{T}}_r & \underline{\underline{I}}_r \end{array} \right] \tag{3-47}$$

$$\hat{\underline{\underline{\theta}}}(T) = \left[\begin{array}{c} \underline{\underline{\theta}}(T) \\ \hline \underline{\underline{0}} \end{array} \right] \tag{3-48}$$

$$\hat{\underline{\underline{\Delta}}}(T) = \left[\begin{array}{c} \underline{\underline{\Delta}}(T) \\ \hline \underline{\underline{0}} \end{array} \right] \quad . \tag{3-49}$$

It is observed that in these three alternative discrete representations of the augmented system, the augmented system matrix $\hat{\underline{\underline{\phi}}}(T)$ has the same set of eigenvalues. Newell and Fisher's and

Porter and Crossley's expressions for $\underline{z}(kT)$ can be interpreted as difference approximations of Equation (3-1). The only difference in these two representations is that a backward difference equation is used by Newell and Fisher while Porter and Crossley used a forward difference equation in discretizing Equation (3-1). Obviously, the accuracy of these representations will depend upon the magnitude of the sampling time, T, used. For a sufficiently small sampling time, Porter and Corssley's representation will give the simplest version of the continuous augmented system. However, in this study, Newell's representation has been adopted since this approach was experimentally tested.

Discrete PI Controller Design

It is clear from Equation (3-42) that the eigenvalues of the augmented discrete system is the union of the eigenvalues of the $n \times n$ matrix $\underline{\underline{\phi}}(T)$ and the $r \times r$ identity matrix $\underline{\underline{I}}_r$. For satisfactory PI control, the r unity eigenvalues should be moved to inside of the unit circle in the complex plane by employing a discrete control law of the form:

$$\underline{u}(kT) = [\underline{\underline{K}}_{FB} \mid \underline{\underline{K}}_I]\hat{\underline{x}}(kT) \quad . \tag{3-50}$$

The complete state controllability of the system given by Equation (3-35) is a necessary and sufficient condition for arbitrary eigenvalue assignment in view of the relation between state controllability and eigenvalue assignability of a system [41, 42, 49]. These controllability conditions for the augmented discrete system of Equation (3-35) can be stated as:

i) the pair $[\underline{\underline{\phi}}(T), \underline{\underline{\Delta}}(T)]$ is a controllable pair,

ii) $$\text{rank} \left[\begin{array}{c|c} \underline{\underline{\phi}}(T)-\underline{\underline{I}}_n & \underline{\underline{\Delta}}(T) \\ \hline \underline{\underline{T}}_r & \underline{\underline{0}} \end{array} \right] = n+r \quad .$$

This can be proved by using the argument given by Davison and Smith [7] for continuous systems.

If the augmented system of Equation (3-35) satisfies these two conditions, the proposition and the two algorithms described in Section 3.3.1 can be extended to shift as many of the $n+r$ eigenvalues as desired, to arbitrary specified positions. Due to the difference in the structure of the augmented system matrices for continuous and discrete systems, some modifications are necessary for two design algorithms to be applicable to the augmented discrete system. Only those modifications, other than simple substitution of $\underline{\underline{\phi}}(T)$ and $\underline{\underline{\Delta}}(T)$ for $\underline{\underline{A}}$, $\underline{\underline{B}}$, are described below.

For discrete systems, Equation (3-15) should be modified to

$$\hat{\underline{\underline{V}}}^T = \left[\begin{array}{c|c} \underline{\underline{V}}^T & \underline{\underline{0}} \\ \hline -T\underline{\underline{T}}_r\{\underline{\underline{I}}_n+[\underline{\underline{\phi}}(T)+\underline{\underline{\Delta}}(T)\underline{\underline{K}}-\underline{\underline{I}}_n]^{-1}\} & \underline{\underline{I}}_r \end{array} \right] \quad . \tag{3-51}$$

By performing a linear transformation on Equation (3-35), $\hat{\underline{y}}(kT) = \hat{\underline{\underline{V}}}^T\hat{\underline{x}}(kT)$, a Jordan canonical form similar to Equation (3-16) is obtained with the identity matrix $\underline{\underline{I}}_r$ in the lower diagonal block and Equation (3-18) becomes

$$\underline{\underline{H}}_2 = -T\underline{\underline{T}}_r[\underline{\underline{\phi}}(T)+\underline{\underline{\Delta}}(T)\underline{\underline{K}}-\underline{\underline{I}}_n]^{-1}\,\underline{\underline{\Delta}}(T) \quad . \tag{3-52}$$

With this modification of matrix $\underline{\underline{H}}_2$, Equation (3-29) becomes

$$\delta_i = \frac{\rho_{n+i}^{-1}}{\langle \underline{\underline{H}}_2 \underline{g}_i ,\ \underline{v}_i^{(i-1)} \rangle} \quad . \tag{3-53}$$

The location of the r eigenvalues which are moved in Step 2) of the Simultaneous Design is determined by the roots of $\det [\lambda \underline{\underline{I}}_r - \underline{\underline{I}}_r - \underline{\underline{H}}_2 \underline{\underline{K}}_I] = 0$. Hence, the necessary modification of Equation (3-32) is given by

$$\underline{\underline{H}}_2 \underline{\underline{K}}_I + \underline{\underline{I}}_r = \underline{\underline{D}} \quad . \tag{3-54}$$

Then, clearly the integral controller matrix $\underline{\underline{K}}_I$ is represented by Equation (3-34) becomes

$$\underline{\underline{K}}_I = \underline{\underline{H}}_2^T (\underline{\underline{H}}_2 \underline{\underline{H}}_2^T)^{-1} (\underline{\underline{D}} - \underline{\underline{I}}_r) \quad . \tag{3-55}$$

Finally, the closed-form of the feedback PI controller matrix is given explicitly by

$$\hat{\underline{\underline{K}}} = [\underline{\underline{K}} - T \underline{\underline{K}}_I \underline{\underline{T}}_r \{ \underline{\underline{I}}_n + [\underline{\underline{\phi}}(T) + \underline{\underline{\Delta}}(T) \underline{\underline{K}} - \underline{\underline{I}}_n]^{-1} \} \ \vdots \ \underline{\underline{K}}_I] \quad . \tag{3-56}$$

Equation (3-50) is the discrete equivalent of Equation (3-27) for the continuous augmented system.

It is emphasized that the general procedure which is described in Section 3.3.1 is applicable to both continuous and discrete systems subject only to controllability conditions on the resulting augmented system. The difference between the continuous and discrete augmented systems arises from the fact that the r repeated zero eigenvalues in the continuous-time domain map into r unity eigenvalues in the discrete-time domain.

CHAPTER FOUR

SIMULATION RESULTS FOR A DOUBLE EFFECT EVAPORATOR

4.1 INTRODUCTION

This chapter presents simulation studies of the eigenvalue assignment techniques presented in Chapters Two and Three applied to a model of the pilot plant double effect evaporator in this department. A 5-state linear discrete model is used in the simulation studies. The simulation results are divided into two main parts: proportional and PI control of the process, and are presented in this order. Before designing the controllers, a modal analysis of the evaporator model is performed to reveal the modal characteristics of the process model.

The first part of the simulation study concerns the design of the proportional controllers and also discusses the performance of the resulting closed-loop systems. The recursive design technique described in Chapter Two is used and the open-loop eigenvalues are moved one by one. This control technique was selected for three reasons: 1) the evaporator model is "derogatory" as shown in Section 4.3, 2) eigenvectors at each recursive stage can be easily updated following the procedure outlined in Chapter Two, and 3) a better insight into the resulting process dynamics and controller gains can be attained by changing eigenvalues in a sequential manner.

In the second part, PI controllers are designed such that eigenvalues of the original system are moved first, and then the repeated unity eigenvalues introduced by integral action are shifted using the methods presented in Chapter Three.

The performance of the controllers derived from the eigenvalue assignment technique is compared with that obtained from optimal control theory to reveal the characteristics of the eigenvalue assignment techniques. Applications of optimal control theory to the evaporator were investigated extensively by Newell [22] using the five state linear model. The optimal feedback control laws used by Newell were derived using dynamic programming and minimizing a suitably chosen quadratic performance index. The discrete optimal feedback control laws used in this chapter were calculated in a similar fashion using the computer program, GEMSCOPE [47].

Since the eigenvalue assignment techniques dealt with in this thesis use the modal information of the system, it was necessary to accurately calculate the eigenvalues and eigenvectors of the open-loop system. At the present time the QR double iterative method seems to give the most accurate eigenvalues and eigenvectors of a general square matrix [45]. A program, CS201A [20], based on this QR double iterative method was used in this investigation.

4.2 MATHEMATICAL MODEL

A simplified schematic diagram of the pilot plant double effect evaporator is shown in Figure 4.1. The first effect is a natural circulation calandria type unit with a nominal feed rate of 5 lb/min of three percent triethylene glycol by weight. This feed is heated by a nominal 2 lb/min of fresh steam. The second effect is an externally forced-circulation long-tube vertical unit with three 1" × 6' tubes and it utilized the vapor from the first effect as its heating medium. The vapor from the second effect is totally

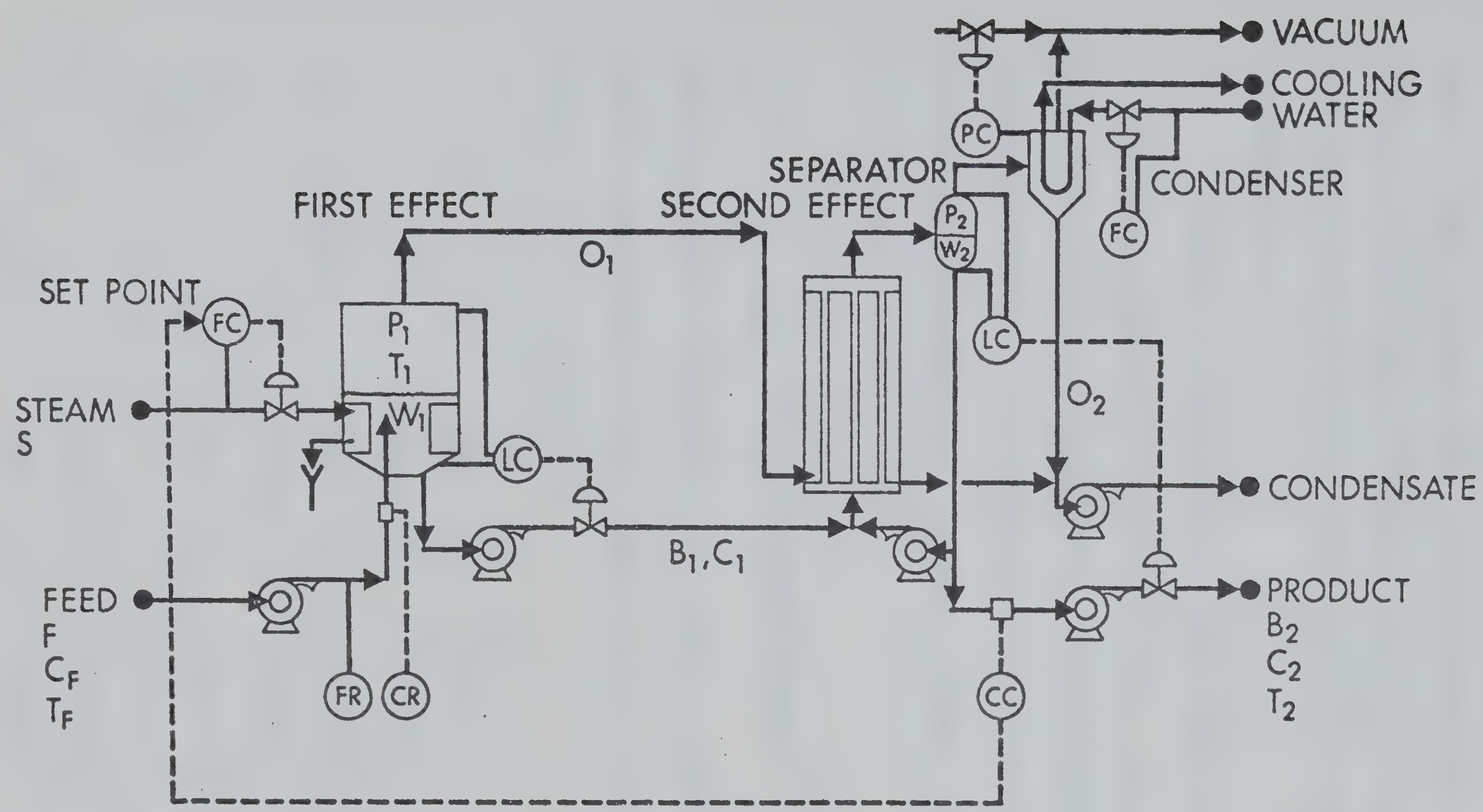

Figure 4.1 Schematic Diagram of Double Effect Evaporator

condensed and the final product is about 1.5 lb/min of ten percent glycol. Tight vacuum control on the second effect maintains the necessary temperature differential across the tube wall. A more detailed description of the evaporator can be found in the thesis by Newell [22].

Models of the evaporator have been extensively investigated by Andre [2], Newell [22] and Wilson [48]. The model used in this thesis was derived by Wilson [48] by linearizing the nonlinear model. The process variables are employed in normalized perturbation form and the resulting model is a 5-state, linear, time-invariant model in the form of Equation (2-4).

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\underline{x}(t)+\underline{\underline{B}}\underline{u}(t)+\underline{\underline{D}}\underline{d}(t) \quad . \tag{2-4}$$

Numerical values of matrices $\underline{\underline{A}}$, $\underline{\underline{B}}$ and $\underline{\underline{D}}$ can be found in Wilson [48] and the state vector $\underline{x}(t)$, control vector $\underline{u}(t)$, and disturbance vector $\underline{d}(t)$ are defined as:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = \begin{bmatrix} W_1'(t) \\ C_1'(t) \\ h_1'(t) \\ W_2'(t) \\ C_2'(t) \end{bmatrix}$$

$$\underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} S'(t) \\ B_1'(t) \\ B_2'(t) \end{bmatrix} \tag{4-1}$$

$$\underline{d}(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \end{bmatrix} = \begin{bmatrix} F'(t) \\ C_F'(t) \\ h_F'(t) \end{bmatrix}$$

where W_1', C_1', S, etc. denote normalized perturbation variable,

e.g. $$W_1 = \frac{(W_1 - W_{1ss})}{W_{1ss}} \qquad \text{(ss = steady state)}$$

and the symbols are defined in the Nomenclature section.

For the digital algorithms used in this study, the continuous model of the evaporator was discretized using the analytical solution to give the discrete model of the form of Equation (2-33), e.g.,

$$\underline{\underline{x}}[(k+1)T] = \underline{\underline{\phi}}(T)\underline{x}(kT) + \underline{\underline{\Delta}}(T)\underline{u}(kt) + \underline{\underline{\theta}}(T)\underline{d}(kT) \quad . \tag{2-33}$$

The numerical values of matrices $\underline{\underline{\phi}}(T)$, $\underline{\underline{\Delta}}(T)$ and $\underline{\underline{\theta}}(T)$ are given in Table 4.1 for a sampling time T of 64 seconds.

To design a PI controller for the evaporator using eigenvalue assignment techniques, the matrix $\underline{\underline{T}}_r$ in Equation (3-1) must be selected properly. At most 3 states can be selected for integral control due to the rank condition given in Section 3.3. Although arbitrary selection of the three variables is possible from among the five state variables, the holdups of the first and second effects W_1 and W_2, and the product concentration, C_2, were chosen based on the physical importance of regulating these three variables. Consequently, the numerical value of matrix $\underline{\underline{T}}_r$ defined in Equation

TABLE 4.1

Numerical Values of Matrices $\underline{\underline{\phi}}(T)$, $\underline{\underline{\Delta}}(T)$ and $\underline{\underline{\theta}}(T)$

T = 64 seconds

$$\underline{\underline{\phi}}(T) = \begin{bmatrix} 1.0 & -0.0008 & -0.0912 & 0.0 & 0.0 \\ 0.0 & 0.9223 & 0.0871 & 0.0 & 0.0 \\ 0.0 & -0.0042 & 0.4377 & 0.0 & 0.0 \\ 0.0 & -0.0009 & -0.1051 & 1.0 & 0.0001 \\ 0.0 & 0.0391 & 0.1048 & 0.0 & 0.9603 \end{bmatrix}$$

$$\underline{\underline{\Delta}}(T) = \begin{bmatrix} -0.0119 & -0.0817 & 0.0 \\ 0.0116 & 0.0 & 0.0 \\ 0.1568 & 0.0 & 0.0 \\ -0.0137 & 0.0847 & -0.0406 \\ 0.0137 & -0.0432 & 0.0 \end{bmatrix}$$

$$\underline{\underline{\theta}}(T) = \begin{bmatrix} 0.1181 & 0.0 & -0.0050 \\ -0.0351 & 0.0785 & 0.0049 \\ -0.0136 & -0.0002 & 0.0662 \\ 0.0012 & 0.0 & -0.0058 \\ -0.0019 & 0.0016 & 0.0058 \end{bmatrix}$$

(3-1) is given by

$$\underline{\underline{T}}_r = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{4-2}$$

The continuous-time augmented system can be formulated directly by combining Equations (2-4) and (4-1) and results in a eight-state variable model. As shown in Section 3.3.2, the discrete augmented model of the evaporator is expressed in the form of Equation (3-35)

$$\hat{\underline{x}}[(k+1)T] = \hat{\underline{\underline{\phi}}}(T)\hat{\underline{x}}(kT) + \hat{\underline{\underline{\Delta}}}(T)\underline{u}(kT) + \hat{\underline{\underline{\theta}}}(T)\underline{d}(kT) \quad . \tag{3-35}$$

Although several approaches can be employed for the calculation of numerical values of matrices $\hat{\underline{\underline{\phi}}}(T)$, $\hat{\underline{\underline{\Delta}}}(T)$ and $\hat{\underline{\underline{\theta}}}(T)$, Newell and Fisher's approach was used in this study. Numerical values of matrices $\hat{\underline{\underline{\phi}}}(T)$, $\hat{\underline{\underline{\Delta}}}(T)$ and $\hat{\underline{\underline{\theta}}}(T)$ are given in Table 4.2.

4.3 MODAL CHARACTERISTICS OF THE EVAPORATOR MODEL

The first step in the application of eigenvalue assignment techniques presented in Chapters Two and Three to the process control is to calculate eigenvalues and eigenvectors of the open-loop system model. For the discrete evaporator model, a modal analysis of the open-loop system gives:

Eigenvalues:

$$\{\lambda_i\} = \{\lambda_1,\ldots,\lambda_5\} = \{1.0, \ 0.9215, \ 0.4385, \ 1.0, \ 0.9603\}$$

TABLE 4.2

Numerical Values of Matrices $\hat{\underline{\underline{\phi}}}(T)$, $\hat{\underline{\underline{\theta}}}(T)$ and $\hat{\underline{\underline{\Delta}}}(T)$

$$
\hat{\underline{\underline{\phi}}}(T) = \left[\begin{array}{ccccc:ccc}
1.0 & -0.0008 & -0.0912 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.9223 & 0.0871 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.0042 & 0.4377 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.0009 & -0.1051 & 1.0 & 0.0001 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0391 & 0.1048 & 0.0 & 0.9603 & 0.0 & 0.0 & 0.0 \\
\hdashline
1.0667 & -0.0008 & -0.0972 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
0.0 & -0.0009 & -0.1121 & 1.0667 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0417 & 0.1118 & 0.0 & 1.0243 & 0.0 & 0.0 & 1.0
\end{array}\right]
$$

Table 4.2 - continued

$$\hat{\underline{\underline{\Delta}}}(T) = \left[\begin{array}{rrr} -0.0119 & -0.0817 & 0.0 \\ 0.0116 & 0.0 & 0.0 \\ 0.1568 & 0.0 & 0.0 \\ -0.0137 & 0.0847 & -0.0406 \\ 0.0137 & -0.0432 & 0.0 \\ \hdashline -0.0127 & -0.0871 & 0.0 \\ -0.0147 & 0.0904 & -0.0433 \\ 0.0146 & -0.0461 & 0.0 \end{array}\right] \qquad \hat{\underline{\underline{\theta}}}(T) = \left[\begin{array}{rrr} 0.1181 & 0.0 & -0.0050 \\ -0.0351 & 0.0785 & 0.0049 \\ -0.0135 & -0.0002 & 0.0662 \\ 0.0012 & 0.0 & -0.0058 \\ -0.0019 & 0.0016 & 0.0058 \\ \hdashline 0.1260 & 0.0 & -0.0054 \\ 0.0013 & 0.0 & -0.0062 \\ -0.0020 & 0.0017 & 0.0062 \end{array}\right]$$

Left Eigenvector Matrix:

$$\underline{\underline{V}} = [\underline{v}_1,\ldots,\underline{v}_5] = \begin{bmatrix} 0.9871 & 0.0 & 0.0 & 0.6866 & 0.0 \\ -0.0014 & 0.9842 & 0.0087 & -0.0006 & 0.6804 \\ -0.1602 & 0.1771 & 0.9999 & -0.2393 & 0.2514 \\ 0.0 & 0.0 & 0.0 & 0.6866 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0023 & 0.6884 \end{bmatrix}$$

Right Eigenvector Matrix:

$$\underline{\underline{W}} = [\underline{w}_1,\ldots,\underline{w}_5] = \begin{bmatrix} 1.013 & 0.0 & 0.1623 & 0.0 & 0.0 \\ 0.0 & 1.0176 & -0.1802 & 0.0 & 0.0 \\ 0.0 & -0.0088 & 1.0015 & 0.0 & 0.0 \\ -1.013 & 0.0012 & 0.1872 & 1.4565 & -0.0048 \\ 0.0 & -1.0026 & 0.1876 & 0.0 & 1.4553 \end{bmatrix}$$

The above numbering of eigenvalues and eigenvectors is arbitrary, but convenient for the treatment that follows. However, vectors $\underline{v}_i$ and $\underline{w}_i$ in matrices $\underline{\underline{V}}$ and $\underline{\underline{W}}$ are the left and right eigenvectors associated with the i^{th} eigenvalue λ_i, respectively. The elements of vector, $\underline{v}_i$ are normalized such that $<\underline{v}_i,\underline{v}_i> = 1$, $<\underline{v}_i\underline{w}_i> = 1$ and $<\underline{v}_i,\underline{w}_j> = 0 \quad (i \neq j)$.

It is clear that the evaporator model has two repeated eigenvalues of unity, which result in unstable open-loop responses for step changes in some of the inputs. Furthermore, the eigenvectors associated with these unity eigenvalues are linearly independent, thus the nature of the system is derogatory. It should be noted that eigenvectors associated with the repeated eigenvalues are not unique, since

any vector resulting from a linear combination of these two eigenvectors also satisfies the definition of the eigenvector. However, the resulting two eigenvectors must be linearly independent.

The diagonal elements of matrices $\underline{\underline{V}}$ and $\underline{\underline{W}}$ are much larger in magnitude than the off-diagonal elements except for two columns, where one other element is comparable to the diagonal element. Since matrices $\underline{\underline{\phi}}(T)$ and $\underline{\underline{W}}$ are diagonally dominant, it is possible to associate each eigenvalue λ_i with a single state variable of the system, i.e., the first eigenvalue λ_1 is associated with the first state x_1, etc. The structure of the right eigenvectors reveals an interesting fact that the three states corresponding to the stable eigenvalues of the system are completely decoupled from the two unstable eigenvalues, hence these three states will be stabilized in the presence of sustained disturbance. The two unstable eigenvalues interact in such a way that, in the presence of sustained disturbances, the first state W_1 will exhibit an unbounded response independent of the fourth state W_2, but the effect of a disturbance on the latter may be compensated partially by the interaction from the eigenvalue associated with the first state. This interaction of the two unstable eigenvalues on the fourth state will depend upon the type of disturbance.

Figure 4.2 shows the simulated open-loop response of four state variables to a step disturbance of 20% in feed flow rate. The reasoning mentioned above explains the slow increase in the second effect level in contrast with the rapid, unstable response of the first effect level. The other two states, the first effect con-

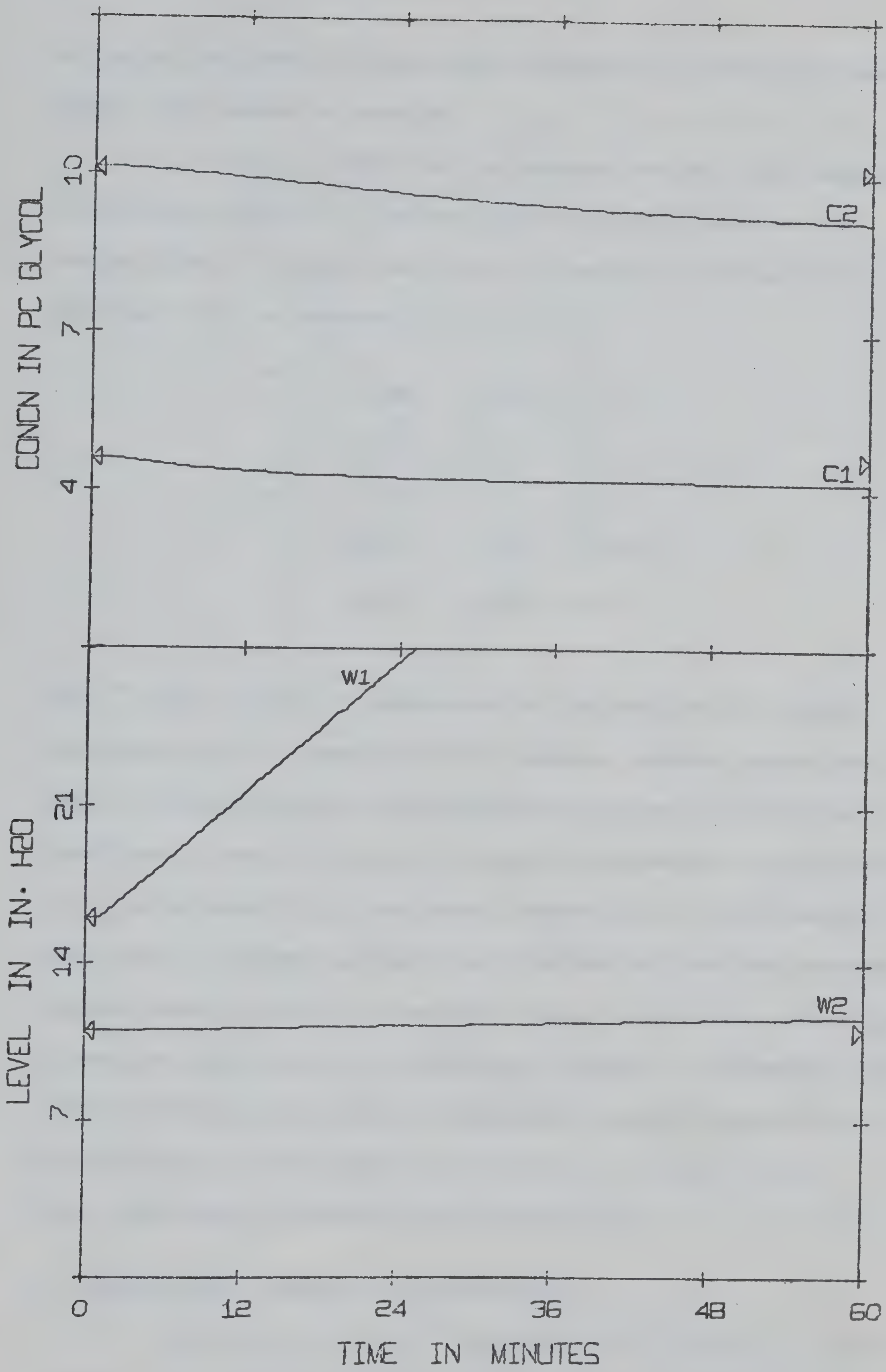

Figure 4.2 Simulated Open-Loop Response for a Step Change of +20% in Feed Flow Rate

centration and product concentration, eventually settle out to new steady state values, as expected.

In order to stablilize the process, the two unity eigenvalues must be moved to locations inside the unit circle in the complex plane. The mode controllability matrix $\underline{\underline{H}}$ defined by Equation (2-10) was calculated to be,

$$\underline{\underline{H}} = \begin{bmatrix} -0.0369 & -0.0806 & 0.0 \\ 0.0392 & 0.0 & 0.0 \\ 0.1569 & 0.0 & 0.0 \\ -0.0551 & 0.0020 & -0.0279 \\ 0.0567 & -0.0298 & 0.0 \end{bmatrix} \tag{4-3}$$

The i^{th} row of matrix $\underline{\underline{H}}$ represents the influence of the control vector upon the i^{th} eigenvalue of the system. However, since the system is derogatory with two repeated eigenvalues, the first and fourth rows of matrix $\underline{\underline{H}}$ must be linearly independent for the system to be controllable [41]. The linear independence of the first and fourth rows is evident and the other rows have at least one non-zero element; thus the system is completely state controllable and all the eigenvalues can be shifted to arbitrary locations. It should be kept in mind that these two repeated eigenvalues cannot be changed by the simultaneous design mentioned in Section 2.3.2 and thus, at least, two stages of recursion are required [41].

4.4 PROPORTIONAL CONTROL OF THE EVAPORATOR

The simulation results presented in this section include the

proportional controllers designed by assigning selected eigenvalues to the closed-loop system and also include the responses of the closed-loop system. Several responses are shown in the same figure in order to illustrate the effect of a particular design option upon the dynamics of the closed-loop system. Comparison of the performance of the controllers is mainly based on the magnitude of the gain elements and the dynamic responses of the closed-loop systems to typical step disturbances. In view of their physical importance in maintaining satisfactory control of the evaporator, the dynamic behavior of the first and second effect levels and the product concentration are emphasized.

4.4.1 Separate Use of the Controls

As mentioned in Section 2.3.3, the controls can be used separately to shift eigenvalues by specifying only one non-zero element in the vector $\underline{g}_i$ at each recursive stage. This results in the pairing of open-loop eigenvalues with the controls.

For a multi-input system, the control matrix which assigns the desired eigenvalues to the closed-loop system is not unique and, in general, depends on:

i) the pairing of open-loop eigenvalues with the controls,

ii) the sequence in which open-loop eigenvalues are changed, and

iii) the pairing of open-loop eigenvalues with the desired closed-loop eigenvalues.

The effect of each of these design options is considered below.

Since all five eigenvalues of $\underline{\phi}(T)$ are controllable,

attempts were made to assign the set of desired eigenvalues, $\{\rho_i\} = \{0.1, 0.2, 0.3, 0.4, 0.5\}$, to the closed-loop system.

<u>Effect of Pairing Open-Loop Eigenvalues with Controls</u>

In order to isolate the effect of pairing open-loop eigenvalues with controls, the pairing of open-loop eigenvalues with desired closed-loop eigenvalues and the sequence in which open-loop eigenvalues were shifted, were fixed. Thus, the i^{th} element of the set $\{\lambda_3,\lambda_4,\lambda_1,\lambda_5,\lambda_2\}$ was moved to the i^{th} element of the set $\{0.3, 0.2, 0.1, 0.5, 0.4\}$ at the i^{th} recursive stage.

In Table 4.3, the feedback matrices from Runs 1-3 show the effect of pairing eigenvalues with controls on the magnitude of the controller gain elements. The gain elements depend upon the pairing employed and change drastically from one run to another. In Figures 4.3a and 4.3b, closed-loop responses to a step distrubance of 20% increase in feed flow rate are shown using the three controllers from Runs 1-3. (The symbols used in the figure captions are defined in the Nomenclature section.)

The pairing of eigenvalues with controls used in Run 1 was selected by inspecting the mode controllability matrix $\underline{\underline{H}}$ and deciding which control has the largest influence on a particular eigenvalue. For example, from the numerical values of the elements in matrix $\underline{\underline{H}}$, λ_1 is most influenced by u_2, λ_2 by u_1, and λ_3 by u_1. The other two eigenvalues, λ_4 and λ_5, were paired with u_3 and u_2 to avoid excessive use of u_1. These pairings agreed with physical intuition except for pairing λ_5 with u_2. However, the resulting feedback matrix is not satisfactory since

TABLE 4.3

Design Configuration and Resulting $\underline{\underline{K}}_{FB}$ in the Separate Use of Controls

Run No.	Design Configuration(*)	$\underline{\underline{K}}_{FB}$
1	$\begin{matrix} \lambda_3 & \lambda_4 & \lambda_1 & \lambda_5 & \lambda_2 \\ 0.3 & 0.2 & 0.1 & 0.5 & 0.4 \\ u_1 & u_3 & u_2 & u_2 & u_1 \end{matrix}$	$\begin{bmatrix} 0.0 & -16.87 & -2.96 & 0.0 & 0.0 \\ 138.76 & -227.99 & -74.84 & 0.0 & -230.74 \\ 19.69 & 0.0 & -4.87 & -19.69 & -0.07 \end{bmatrix}$
2	$\begin{matrix} \lambda_3 & \lambda_4 & \lambda_1 & \lambda_5 & \lambda_2 \\ 0.3 & 0.2 & 0.1 & 0.5 & 0.4 \\ u_1 & u_1 & u_2 & u_3 & u_3 \end{matrix}$	$\begin{bmatrix} 12.42 & -0.01 & -3.96 & 12.42 & 0.04 \\ 5.79 & -0.01 & 0.0 & -5.05 & -0.02 \\ -45.20 & -142.44 & 1.17 & 24.17 & 126.14 \end{bmatrix}$
3	$\begin{matrix} \lambda_3 & \lambda_4 & \lambda_1 & \lambda_5 & \lambda_2 \\ 0.3 & 0.2 & 0.1 & 0.5 & 0.4 \\ u_1 & u_1 & u_3 & u_2 & u_1 \end{matrix}$	$\begin{bmatrix} 12.42 & -0.01 & -3.96 & 12.42 & 0.04 \\ -1.24 & -27.81 & 0.22 & 1.45 & 27.89 \\ -25.42 & 0.05 & 0.0 & 22.16 & 0.07 \end{bmatrix}$

Table 4.3 - continued

Run No.	Design Configuration(*)					$\underline{\underline{K}}_{FB}$
	λ_4	λ_1	λ_5	λ_2	λ_3	
4	0.2	0.1	0.5	0.4	0.3	$\begin{bmatrix} 16.69 & 32.03 & -3.85 & 8.21 & -28.70 \\ -1.06 & -26.42 & 0.31 & 1.44 & 27.55 \\ -25.42 & 0.05 & 0.0 & 22.16 & 0.07 \end{bmatrix}$
	u_1	u_3	u_2	u_2	u_1	
	λ_5	λ_4	λ_1	λ_3	λ_2	
5	0.5	0.2	0.1	0.3	0.4	$\begin{bmatrix} 9.19 & -4.61 & -3.99 & 6.29 & -4.64 \\ 71.77 & 196.31 & 0.18 & -8.34 & -129.26 \\ 12.94 & 17.19 & 0.13 & 22.16 & 17.47 \end{bmatrix}$
	u_2	u_3	u_2	u_1	u_1	
	λ_3	λ_4	λ_1	λ_5	λ_2	
6	0.4	0.1	0.2	0.5	0.3	$\begin{bmatrix} 11.98 & -0.01 & -4.03 & 11.98 & 0.04 \\ -1.98 & -32.38 & 0.27 & 1.95 & 32.59 \\ -22.60 & 0.04 & 0.0 & 19.69 & 0.06 \end{bmatrix}$
	u_1	u_1	u_3	u_2	u_2	
	λ_3	λ_4	λ_1	λ_5	λ_2	
7	0.1	0.3	0.2	0.4	0.5	$\begin{bmatrix} 13.98 & -0.09 & -4.33 & 13.98 & 0.04 \\ -1.05 & -27.80 & 0.18 & 1.63 & 27.89 \\ -22.60 & 0.04 & 0.0 & 19.69 & 0.06 \end{bmatrix}$
	u_1	u_1	u_3	u_2	u_2	

*The first row indicates the sequence of changing open-loop eigenvalues and the second and third rows show the corresponding desired closed-loop eigenvalues and controls, respectively.

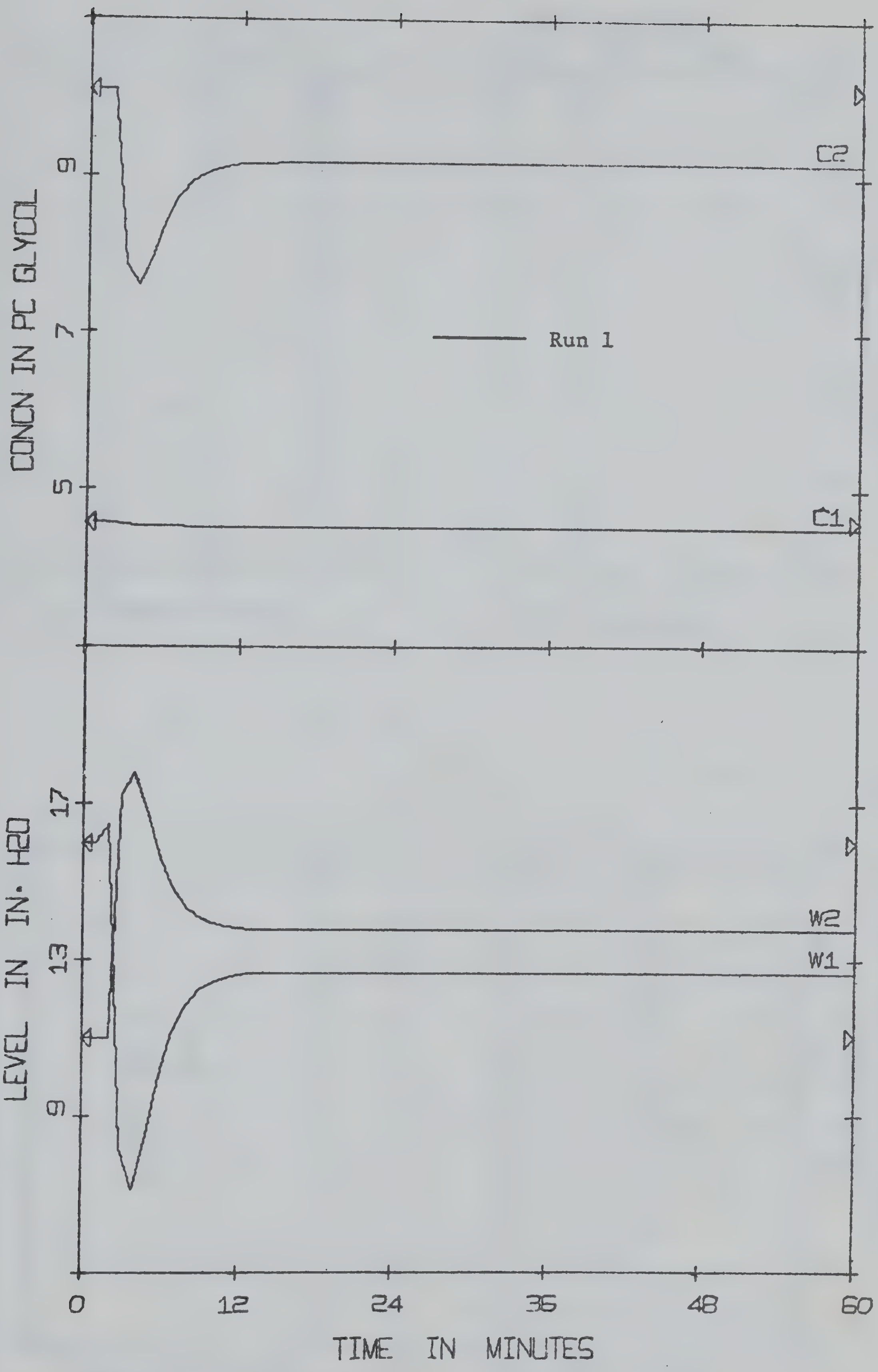

Figure 4.3a Simulated Effect of Pairing λ_i with u_j in the Separate Use of Controls (+20% F/P)

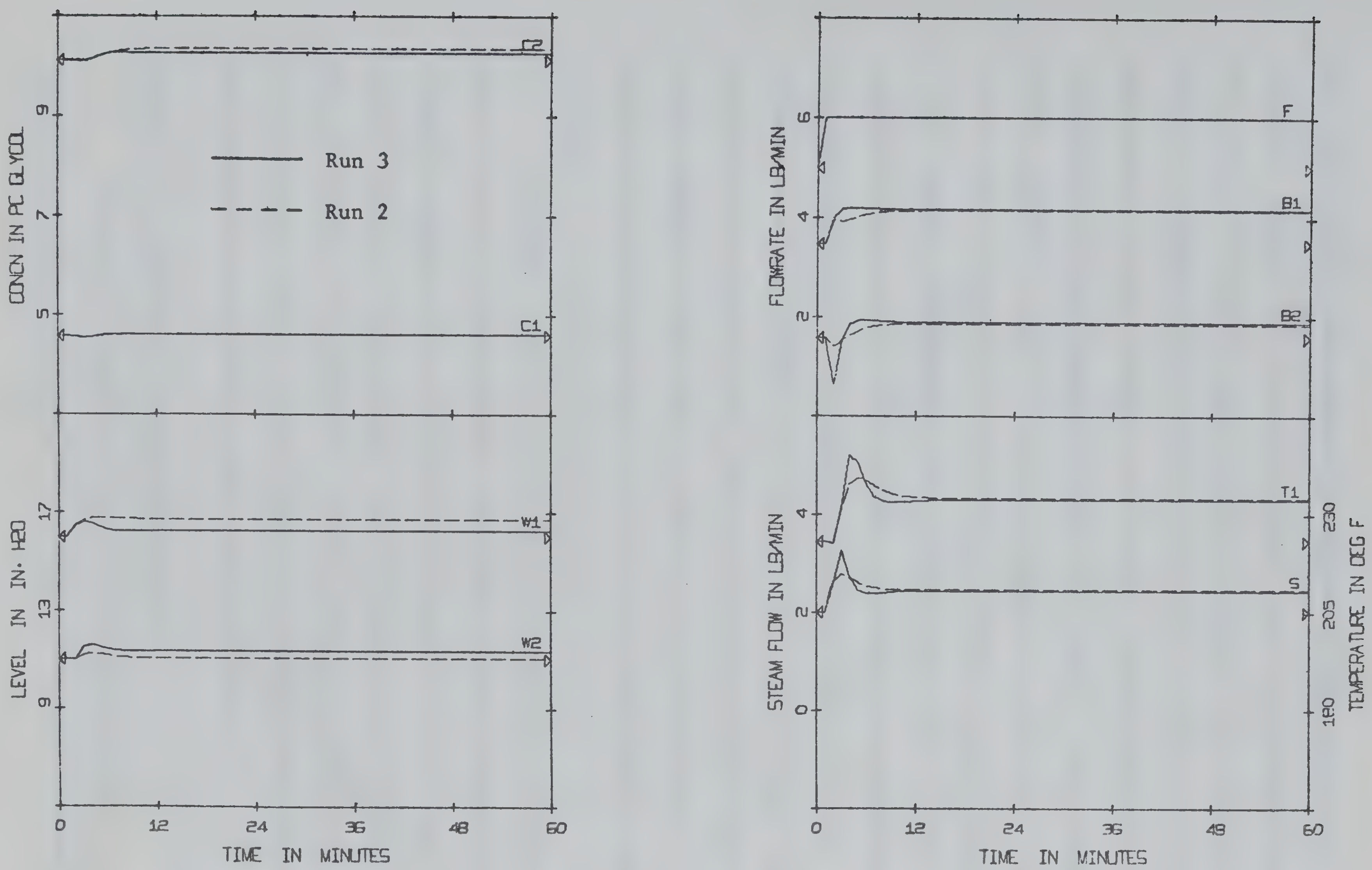

Figure 4.3b Simulated Effect of Pairing λ_i with u_j in the Separate Use of Controls (+20% F/P)

some gain elements are excessively large and a highly undesirable transient results with large steady state errors (cf., Figure 4.3a). The pairing of open-loop eigenvalues with controls used in Runs 2 and 3 was obtained by trial and error in order to ensure that the magnitude of the gain elements was reduced. A drastic reduction in the magnitude of gain elements was achieved in Run 3. The pairings used successfully in Run 3 seem to contradict the influence of controls on the open-loop eigenvalues shown by the numerical values of matrix $\underline{\underline{H}}$. For example, configuration used in Run 3 shows that λ_1 which is associated with the first effect holdup was moved by manipulating the product flow rate. This phenomenon can be explained by the changes in the left eigenvectors associated with the unchanged eigenvalues at each recursive step. Hence, the numerical value of matrix $\underline{\underline{H}}$ also changes at each stage of recursion. In extreme cases, a controllable eigenvalue-control pair at the previous recursion could no longer be controllable at the next stage of recursion.

Figures 4.3a and 4.3b show that if the closed-loop system has the same set of eigenvalues, then a better dynamic response to a feed flow disturbance is obtained as the magnitude of the gain elements decreases. Comparison of the feedback gain matrices for Runs 1-3 reveals the tendency that control effect is mainly achieved by u_2 in Run 1 and u_3 in Run 2, while the control efforts are more uniformly distributed over all controls in Run 3.

Effect of the Sequence of Changing Eigenvalues

In order to study the effect of the sequence used in changing open-loop eigenvalues, the pairing of controls with open-

loop eigenvalues and the pairing of closed-loop and open-loop eigenvalues were fixed. These pairings were the same as used in Run 3. The configurations of Runs 3-5 in Table 4.3 show the effect of the sequence of changing open-loop eigenvalues. The resulting feedback matrices are included to show their structure and the magnitude of the gain elements. The corresponding closed-loop responses to the 20% step increase in feed flow rate are shown in Figure 4.4.

The response of the closed-loop systems in Figure 4.4 shows that if the closed-loop eigenvalues are the same, better dynamic behavior will result for a controller with smaller gain elements. Steady state errors in the state variables due to three different disturbances are shown in Table 4.4 for Runs 3-5. Although, the magnitude of steady state errors varies remarkably from one disturbance to another, the same trend, that the smaller the gain magnitude, the better the dynamic behavior, seems to hold for other disturbances. Furthermore, it is observed from Table 4.4 that the feed concentration disturbance has the most significant effect in comparison with the other disturbances, while the same disturbance has the least effect on the open-loop resonse.

The feedback matrices for Runs 3-5 show that the matnitude of gain elements also depends on the sequence in which open-loop eigenvalues are changed. It is observed from Runs 1-5 that the control effort should be distributed over the available controls to obtain lower gains and good dynamic behavior. It should also be noted that a particular eigenvalue-control pair in one sequence may not be possible in a different sequence of eigenvalue shifts due to a change in left eigenvector directions.

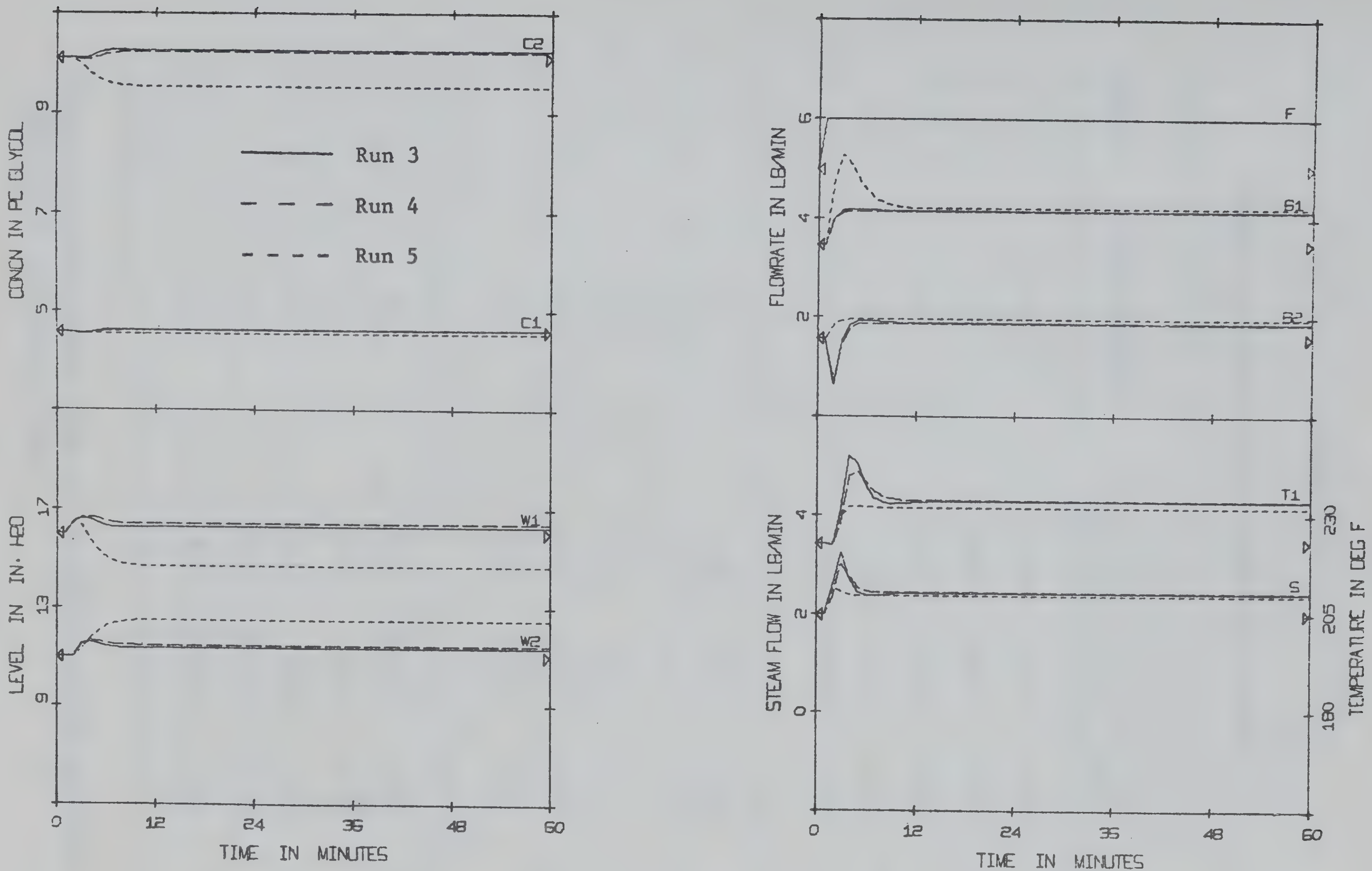

Figure 4.4 Simulated Effect of the Sequence of Changing λ_1 in the Separate Use of Controls (+20% F/P)

TABLE 4.4

Steady State Errors for the Closed-Loop Systems - I

Run No.	State Variables	Steady State Errors(*) to		
		+20% F	-20% CF	+20% TF
3	W_1	1.28	0.01	- 0.32
	C_1	0.61	-19.97	0.0
	h_1	5.68	0.19	0.0
	W_2	2.31	0.11	- 0.36
	C_2	1.20	-19.92	0.01
4	W_1	1.92	3.36	- 0.32
	C_1	0.60	-19.72	0.0
	h_1	5.67	0.31	0.0
	W_2	3.05	3.92	- 0.36
	C_2	1.13	-19.05	0.01
5	W_1	- 6.39	85.80	1.11
	C_1	- 0.98	- 5.20	0.25
	h_1	4.72	9.08	0.15
	W_2	10.25	84.84	- 1.77
	C_2	- 5.86	45.35	1.11

*Steady State Errors were taken at $t = 60$ min and are expressed as a percent of the initial states.

Effect of Pairing Open-Loop and Closed-Loop Eigenvalues

The configurations in Runs 3, 6 and 7 are different only in the pairings of open-loop and desired closed-loop eigenvalues that were used. Although the magnitude of gain elements changes with variations in the open-loop and closed-loop eigenvalue pairings, this effect is small compared with the other factors previously discussed and the resulting feedback matrices have the same general features. The effects of pairing open-loop and closed-loop eigenvalues on the dynamic response are seen in Figure 4.5 for a 20% step disturbance in feed flow rate. The difference in dynamic behavior is very small.

In conclusion, it is observed that the dynamic behavior of a closed-loop system depends highly on the configuration used in shifting open-loop eigenvalues. Of the three options discussed, the sequence of changing eigenvalues and the pairing of open-loop eigenvalues with controls have significant effects on both the gain magnitudes and the closed-loop dynamics. If the closed-loop eigenvalues are the same, the closed-loop response becomes more satisfactory as the gain elements decrease in magnitude. It is interesting that, in the resulting feedback control systems, the holdups of the first and second effects govern the steam (u_1) and product (u_3) flow rates while the bottom flow rate (u_2) from the first effect is governed mainly by the first and second effect concentrations. Furthermore, the controls, in general, are not significantly affected by the enthalpy of the first effect, as indicated by the relatively small elements in the third column of $\underline{\underline{K}}_{FB}$ in Table 4.3.

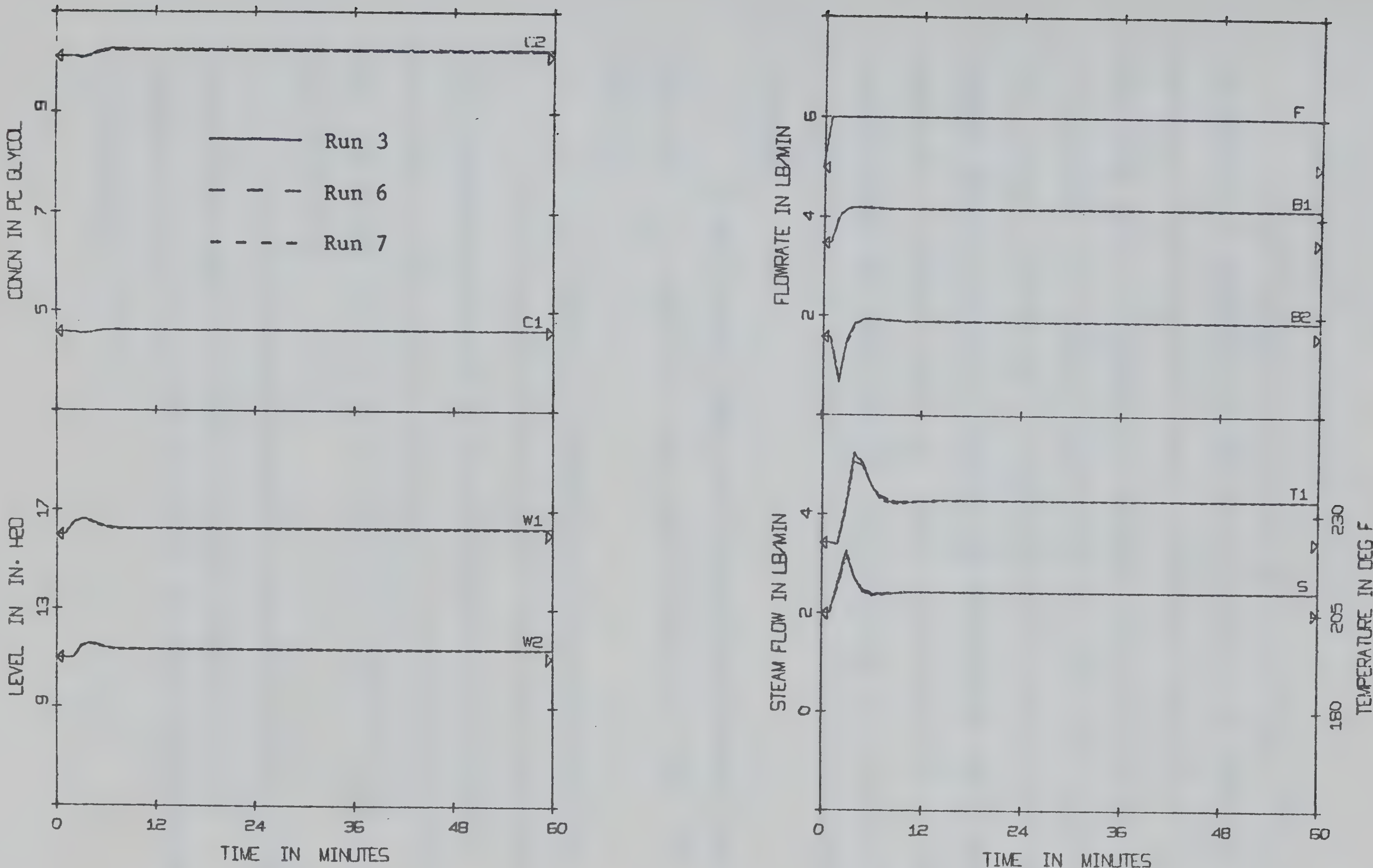

Figure 4.5 Simulated Effect of Pairing λ_i with ρ_i in the Separate Use of Controls (+20% F/P)

4.4.2 Simultaneous Use of Controls

If all the elements in vector $\underline{g}_i$ in Equation (2-21) are specified to be non-zero, then all controls are used to change an eigenvalue at each recursive stage. An arbitrary selection of $\underline{g}_i$, which gives non-zero elements and preserves the controllability of the eigenvalue to be changed, would result in a controller matrix which assigns a desired set of eigenvalues to the closed-loop system. Although Equation (2-32) is valid for gain minimization when a single eigenvalue is changed, vector $\underline{g}_i$ chosen by Equation (2-32) at each recursive stage will guarantee the controllability of an eigenvalue to be shifted but the resulting controller gains will not necessarily be minimal.

In this section, open-loop eigenvalues are changed recursively with $\underline{g}_i$ specified at each stage by Equation (2-32). The effects of

i) the sequence of changing open-loop eigenvalues, and

ii) the pairing of open-loop and closed-loop eigenvalues

on the feedback matrix and the dynamic response are investigated. Furthermore, a controller designed using an arbitrary set of vectors $\{\underline{g}_i\}$, is compared with one of the controllers designed using Equation (2-32).

The set of desired closed-loop eigenvalues chosen in this section is the same as was used in previous subsections. The configuration used in each design and the resulting feedback matrices are summarized in Table 4.5.

TABLE 4.5

Design Configuration and Resulting $\underline{\underline{K}}_{FB}$ in the Simultaneous Use of Controls

Run No.	Design Configuration(*)					$\underline{\underline{K}}_{FB}$
	λ_2	λ_1	λ_3	λ_5	λ_4	
8	0.5	0.1	0.4	0.3	0.2	$\begin{bmatrix} 55.54 & 121.95 & -5.03 & 10.32 & -85.64 \\ 68.83 & 154.46 & -1.37 & 10.32 & -85.64 \\ 11.84 & 18.94 & -0.21 & 10.32 & 1.28 \end{bmatrix}$
	λ_5	λ_4	λ_2	λ_1	λ_3	
9	0.3	0.2	0.5	0.1	0.4	$\begin{bmatrix} 45.67 & 98.12 & -4.81 & -0.45 & -80.67 \\ 45.67 & 108.51 & -0.97 & -0.45 & -70.16 \\ -20.86 & -76.65 & 0.72 & 20.21 & 79.75 \end{bmatrix}$
	λ_1	λ_4	λ_5	λ_2	λ_3	
10	0.1	0.2	0.3	0.5	0.4	$\begin{bmatrix} 28.33 & 37.03 & -3.53 & 9.91 & -20.00 \\ 2.88 & -22.47 & 0.89 & 6.01 & 34.82 \\ -4.19 & -22.45 & -0.13 & 17.65 & 34.86 \end{bmatrix}$

Table 4.5 - continued

Run No.	Design Configuration(*)					$\underline{\underline{K}}_{FB}$
	λ_1	λ_4	λ_5	λ_2	λ_3	
11	0.1	0.2	0.5	0.3	0.4	$\begin{bmatrix} 27.67 & 42.98 & -3.63 & 5.51 & -31.24 \\ -1.32 & -32.84 & 1.05 & 5.60 & 41.57 \\ -8.40 & -32.82 & 0.03 & 17.23 & 41.61 \end{bmatrix}$
	λ_1	λ_4	λ_5	λ_2	λ_3	
12	0.5	0.4	0.3	0.2	0.1	$\begin{bmatrix} 30.97 & 58.30 & -3.74 & 3.13 & -46.06 \\ -6.66 & -44.28 & 2.06 & 12.54 & 60.33 \\ -9.02 & -44.27 & 0.82 & 21.27 & 60.36 \end{bmatrix}$
	λ_1	λ_4	λ_5	λ_2	λ_3	
13	0.1	0.2	0.3	0.5	0.4	$\begin{bmatrix} 43.36 & 93.81 & -5.33 & -5.13 & -86.10 \\ 103.70 & 256.73 & 3.03 & 35.95 & -104.73 \\ 50.16 & 113.13 & -2.28 & 13.44 & -63.88 \end{bmatrix}$

*The first row indicates the sequence of changing open-loop eigenvalues and the second row shows the corresponding desired closed-loop eigenvalues.

Effect of the Sequence of Changing Open-loop Eigenvalues

In Runs 8-10 in Table 4.5 and Figure 4.6, the pairing of open-loop eigenvalues with desired eigenvalues was fixed and the sequence in which open-loop eigenvalues were changed, was varied. As in the case of using controls separately to shift eigenvalues, this effect is significant for both the controller gains and the dynamic behavior of the resulting closed-loop system. It can readily be seen from Figure 4.6 that the controller resulted from Run 10 shows the best response with the smallest offsets in the four state variables. In Table 4.6, steady state errors due to typical step disturbances are given for the closed-loop systems resulted from Runs 8-10. The same dependence of the closed-loop system performance on the type of disturbance as was observed in the separate use of controls can be seen from Table 4.6. Although individual states exhibit different steady state errors depending on the type of disturbance, in general, the dynamic behavior of the closed-loop system is better when the magnitude of the gain elements is smaller for the same closed-loop eigenvalues.

Effect of Pairing Open-Loop and Closed-Loop Eigenvalues

The feedback matrices obtained from Runs 10-12 in Table 4.5 show that the pairing of open-loop and closed-loop eigenvalues can effect the magnitude of the gain elements. Figure 4.7 compares the dynamic response of these three closed-loop systems to the 20% increase in feed flow rate. In contrast with the case of using controls separately, the gain elements change significantly depending on how the open-loop and closed-loop eigenvalues are paired. However, the

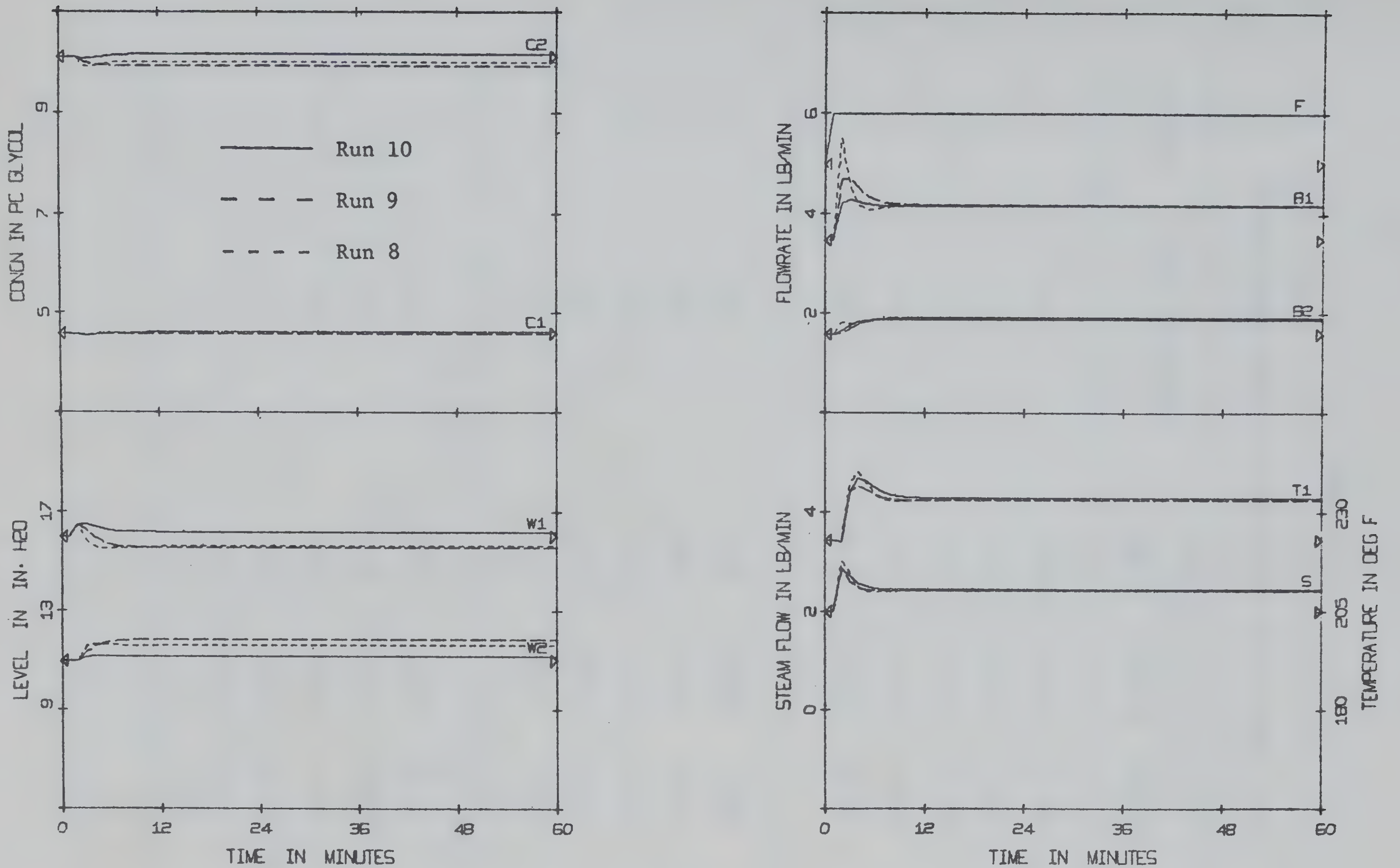

Figure 4.6 Simulated Effect of the Sequence of Changing λ_i in the Simultaneous Use of Controls (+20% F/P)

TABLE 4.6

Steady State Errors for the Closed-Loop Systems - II

Run No	State Variables	Steady State Errors to		
		+20% F	-20% CF	+20% TF
8	W_1	-1.80	36.38	0.38
	C_1	0.10	-15.62	0.09
	h_1	5.37	2.81	0.05
	W_2	4.17	-14.80	-0.69
	C_2	-1.09	- 0.71	0.39
9	W_1	-2.09	45.81	0.58
	C_1	-0.08	-13.30	0.14
	h_1	5.26	4.21	0.08
	W_2	5.87	-42.52	-1.27
	C_2	-1.89	9.57	0.60
10	W_1	0.85	12.06	-0.23
	C_1	0.45	-18.71	0.01
	h_1	5.58	0.95	0.01
	W_2	1.00	7.10	-0.14
	C_2	0.46	-14.36	0.05

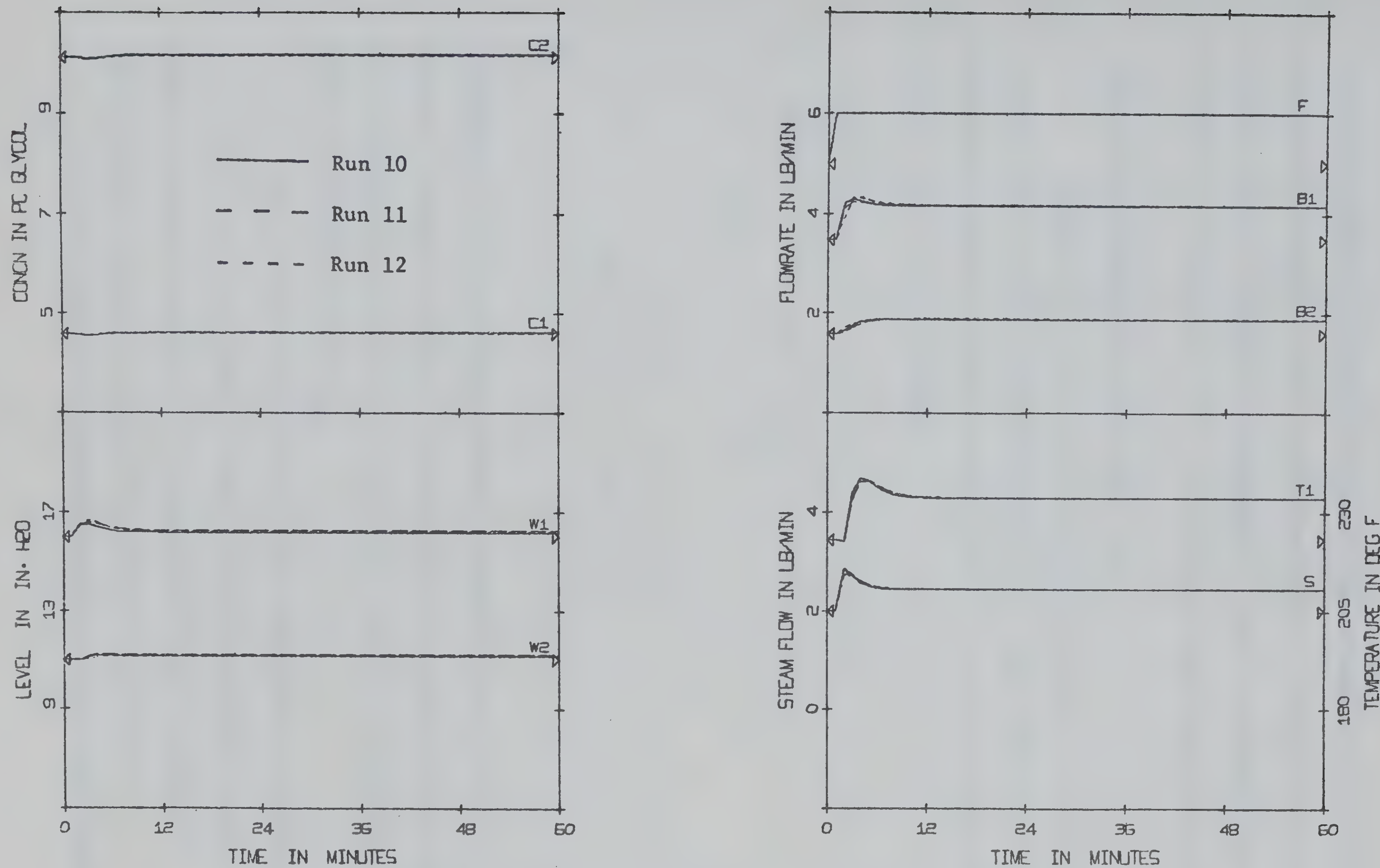

Figure 4.7 Simulated Effect of Pairing λ_i with ρ_i in the Simultaneous Use of Controls (+20% F/P)

controller matrices are similar and the dynamic behavior of the three closed-loop systems is almost indistinguishable. The effect of pairing open-loop and closed-loop eigenvalues is far less significant than the effect of changing the order in which open-loop eigenvalue are shifted.

<u>Effect of the Choice of a Vector Set,</u> $\{\underline{g}_i\}$

In Runs 13 of Table 4.5, the design configuration is identical to that of Run 10 except that the vector set, $\{\underline{g}_i\}$, was specified arbitrarily instead of using Equation (2-32). The resulting set of vectors, $\{\underline{g}_i\}$, are as follows,

$$\underline{g}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{g}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \underline{g}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

$$\underline{g}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \underline{g}_5 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Comparison of the controller matrices from Runs 10 and 13 and the transient response in Figure 4.8 indicates that excessively high gains in the feedback controller can be avoided by making use of Equation (2-32) in the selection of $\underline{g}_i$.

The results of this section can be summarized as follows:

i) As in the case of using controls separately, better dynamic response is expected as the magnitude of gain elements is reduced if the closed-loop system have the same eigenvalues.

ii) The sequence of changing eigenvalues has a more signifi-

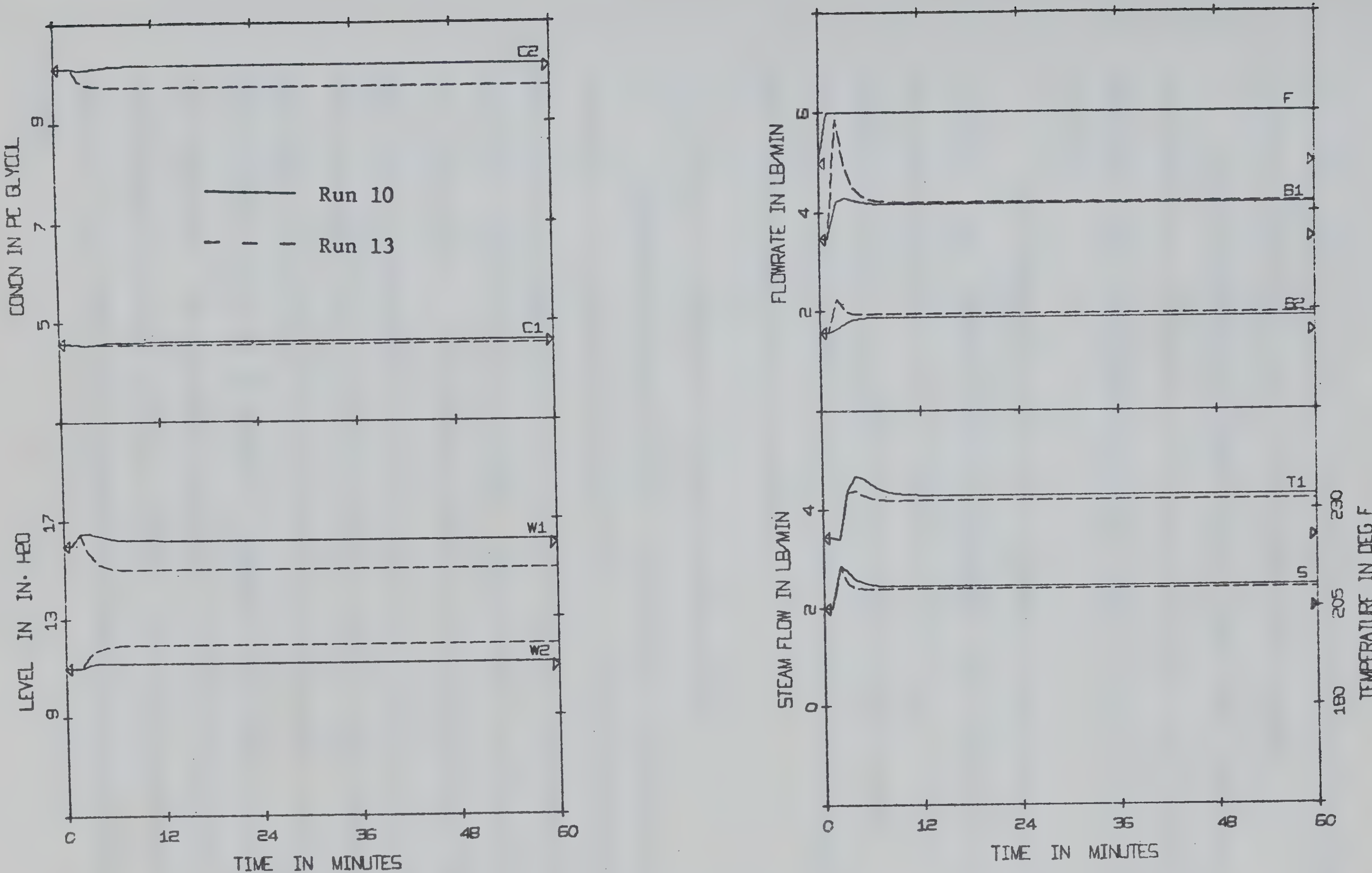

Figure 4.8 Simulated Effect of Choosing g_1 in the Simultaneous Use of Controls (+20% F/P)

cant effect than the pairing of open-loop and closed-loop eigenvalues on both the controller gains and the closed-loop dynamics.

iii) The pairing of open-loop and closed-loop eigenvalues has a significant influence on the controller gains, but the dynamic responses of the resulting closed-loop systems are only slightly affected.

iv) Equation (2-32) can be utilized in specifying a set of vector, $\{\underline{g}_i\}$ in order to avoid extremely large controller gains, which might occur when a set of arbitrary vector $\{\underline{g}_i\}$ are used. Although, each vector $\underline{g}_i$ selected by Equation (2-32) locally minimizes the gain matrix at the i^{th} stage, there is no guarantee that the final gain matrix is minimized.

4.4.3 Effect of the Desired Closed-Loop Eigenvalues

Another important question, independent of the design options discussed in the previous two subsections, is how the magnitude of the closed-loop eigenvalues affects the dynamics of the closed-loop system and the feedback gain matrix. As an attempt to investigate this design choice, two different sets of desired eigenvalues, namely $\{0.2, 0.4, \lambda_3 \text{ (unchanged)}, 0.5, 0.6\}$ and $\{0.4, 0.5, 0.6, 0.7, 0.8\}$, were assigned to the closed-loop systems in Runs 14 and 15 respectively. The design configurations used in Runs 14 and 15 and the resulting gain matrices are given in Table 4.7 together with those of Run 10 for comparison. The simulated responses of the resulting closed-loop systems are shown in Figures 4.9-4.11 for step disturbances in feed flow rate (+20%), feed concentration (-20%) and feed temperature (+20%).

TABLE 4.7

Design Configuration and Resulting $\underline{\underline{K}}_{FB}$ in Assigning Different Set of Eigenvalues

Run No.	Design Configuration					$\underline{\underline{K}}_{FB}$
	λ_1	λ_4	λ_5	λ_2	λ_3	
10	0.1	0.2	0.3	0.5	0.4	$\begin{bmatrix} 28.33 & 37.03 & -3.53 & 9.91 & -20.00 \\ 2.88 & -22.47 & 0.89 & 6.01 & 34.82 \\ -4.19 & -22.45 & -0.13 & 17.65 & 34.86 \end{bmatrix}$
	λ_1	λ_4	λ_5	λ_2	λ_3	
14	0.2	0.4	0.5	0.6	unchanged	$\begin{bmatrix} 19.36 & 20.09 & -2.82 & 7.21 & -10.42 \\ 5.22 & -10.15 & 0.45 & 3.87 & 20.54 \\ -2.17 & -10.13 & 0.03 & 12.60 & 20.51 \end{bmatrix}$
	λ_1	λ_4	λ_5	λ_2	λ_3	
15	0.4	0.5	0.6	0.7	0.8	$\begin{bmatrix} 10.29 & 9.45 & -1.63 & 0.98 & -8.08 \\ 8.45 & -1.85 & -0.20 & 7.63 & 15.82 \\ -3.17 & -5.14 & 0.48 & 4.90 & 9.48 \end{bmatrix}$

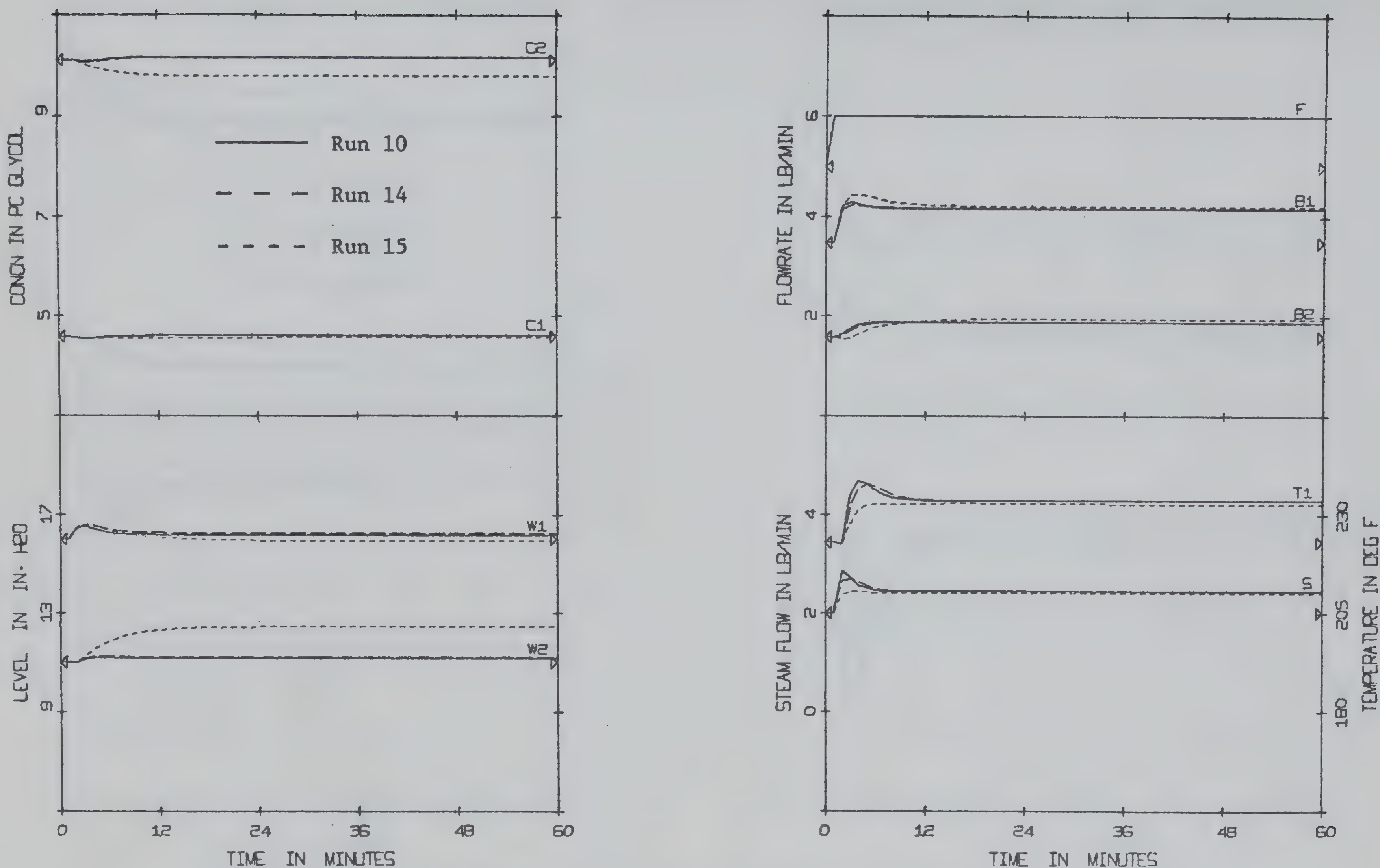

Figure 4.9 Simulated Effect of the Desired Closed-Loop Eigenvalues in the Simultaneous Use of Contols (+20% F/P)

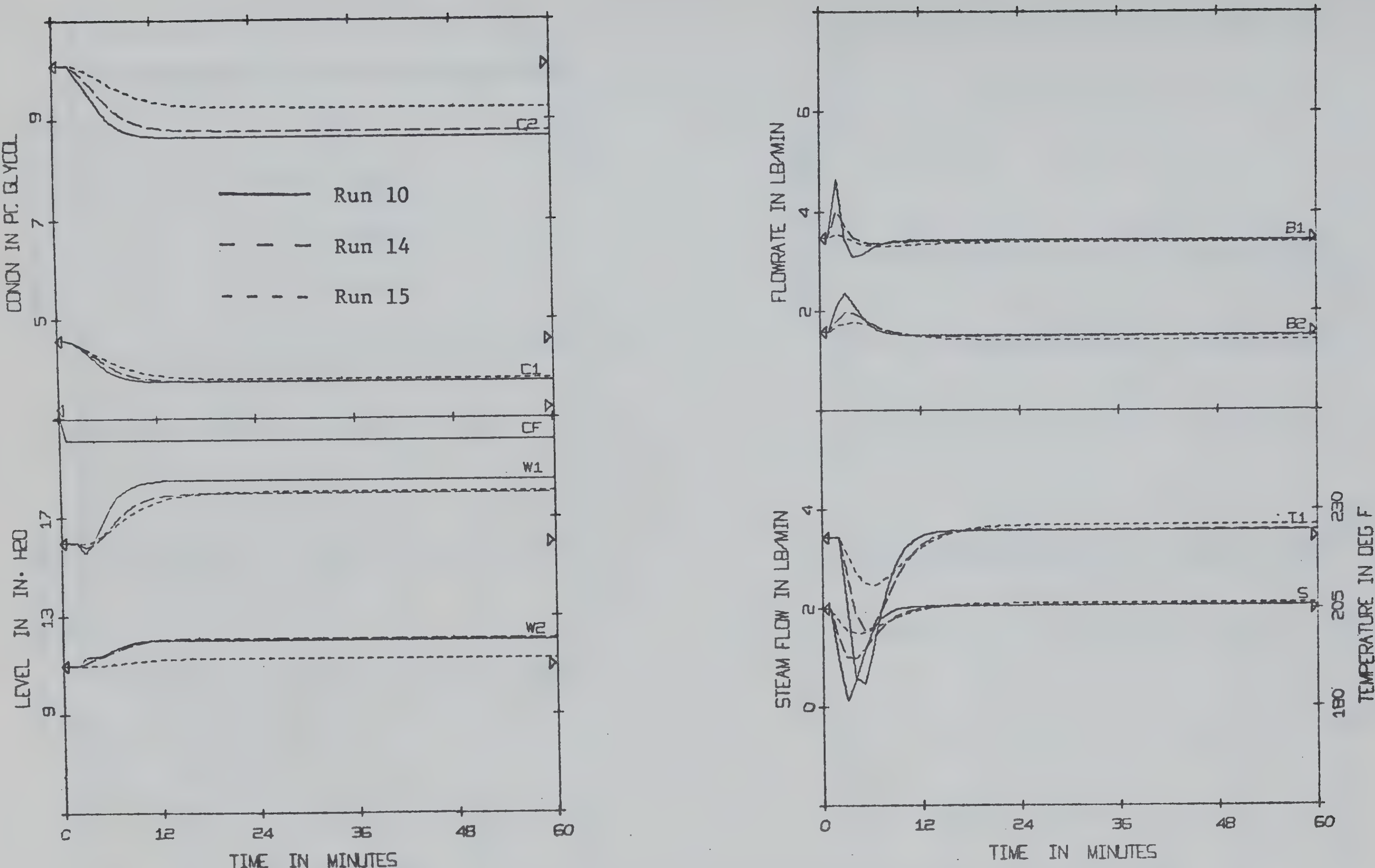

Figure 4.10 Simulated Effect of the Desired Closed-Loop Eigenvalues in the Simultaneous Use of Controls (-20% CF/P)

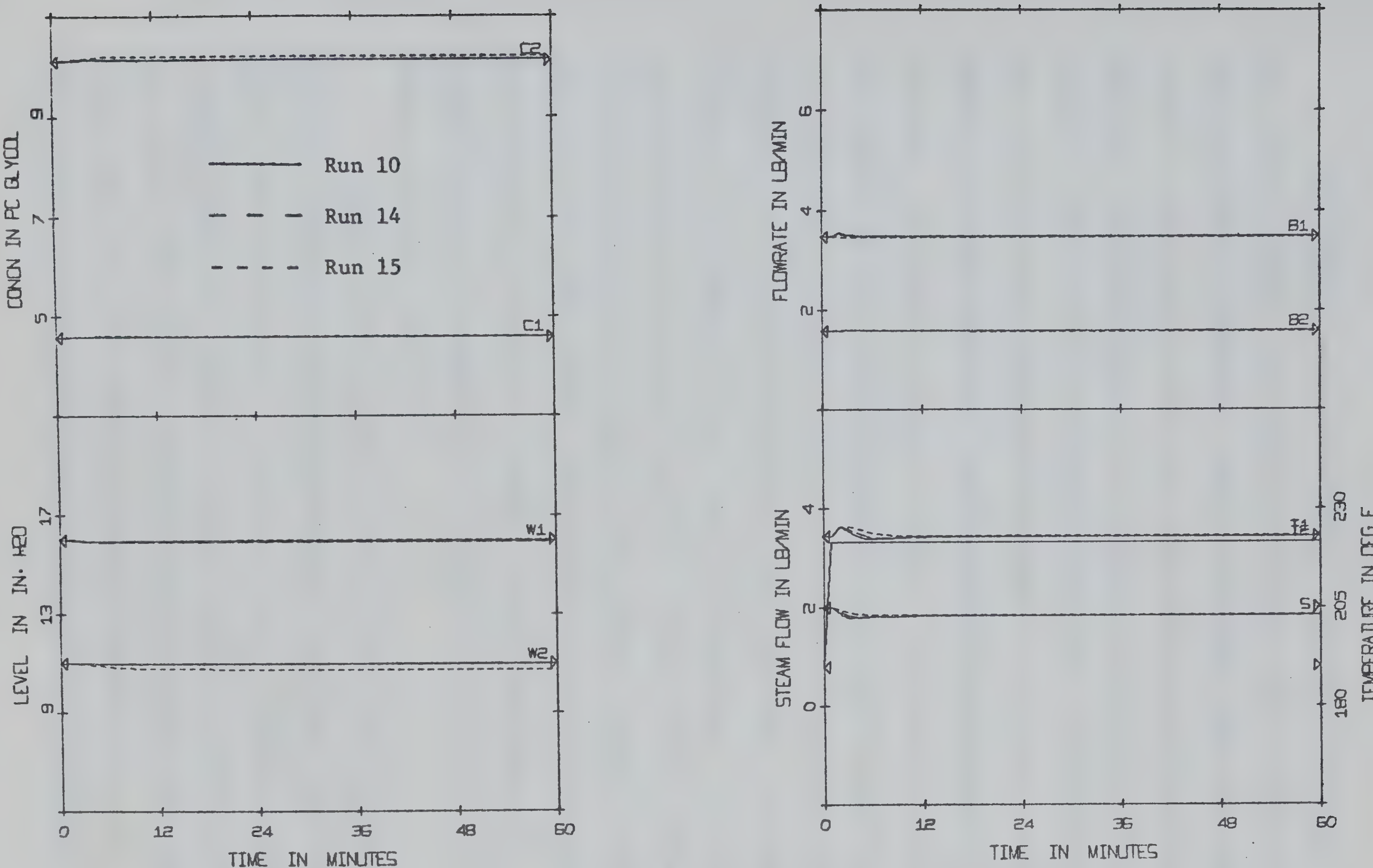

Figure 4.11 Simulated Effect of the Desired Closed-Loop Eigenvalues in the Simultaneous Use of Controls (+20% TF/P)

The feedback matrices in Table 4.7 show that the largest gain element decreases significantly as the magnitude of the closed-loop eigenvalues is increased. This is an expected result since less control will be required as the open-loop eigenvalues are moved smaller distances. The dynamic behavior of the closed-loop system to the disturbance in feed flow rate, as shown in Figure 4.9, is not surprising since the closed-loop eigenvalues and offsets are smallest in Run 10. However, in Figure 4.10 where a feed concentration disturbance is considered, the behavior observed in Figure 4.9 is completely reversed. This behavior is hardly expected judging solely from the mangitude of the closed-loop eigenvalues. The effect of a feed temperature disturbance on the closed-loop dynamics is least significant, as shown in Figure 4.11, compared with the other two disturbances. An interesting observation is that Runs 14 and 10 yield almost identical responses to the step disturbances. Comparing these dynamic responses and the magnitude of the gain elements, the controller from Run 14 performs better than that from Run 10. However, the new steady state is reached faster as the magnitude of the closed-loop eigenvalues is reduced, as would be expected.

From these observations, it is clear that a set of smaller closed-loop eigenvalues does not necessarily yield better performance of the closed-loop system. It is observed that the dynamic response is highly disturbance dependent as was the case in the previous two subsections. This behavior can be explained qualitatively in terms of the directions of the resulting closed-loop eigenvectors. If the resulting closed-loop left eigenvectors happen to be oriented in

the same direction, then from Equation (2-7), it can be shown that the components of the right eigenvectors are very large. Hence, those disturbances whose corresponding coefficient vectors (columns of matrix $\underline{\underline{\theta}}(T)$) are relatively colinear to the left eigenvectors will have significant effects on the dynamic behavior of the state variables, even for small eigenvalues. This point will become more apparent in the next subsection where an optimal control law is compared with control laws derived eigenvalue assignment techniques.

4.4.4 Comparison With an Optimal Controller

Since controllers derived from optimal control theory are not explicitly concerned with shifting eigenvalues and the controllers derived from eigenvalue assignment techniques are not unique for a multi-input system, a meaningful comparison of these two types of controllers is difficult. However, if the closed-loop systems derived from optimal control theory and eigenvalue assignment techniques have the same set of eigenvalues, a comparison of closed-loop dynamics will reveal the effect of the eigenvector orientations of the closed-loop systems.

The optimal control law used in this study was calculated from the GEMSCOPE computer program [47], and is identical to one of the controllers used by Wilson [48]. In view of the physical importance of the state variables to be controlled, the state weighting matrix, $\underline{\underline{Q}} = \text{diag}\ [10, 1, 1, 10, 100]$, used in previous studies [22, 48] was chosen. The control weighting matrix, $\underline{\underline{R}}$, was set equal to the zero matrix. The eigenvalues of the closed-loop system using this optimal control law were calculated to be {0.9002, 0.2706,

5×10^{-6}, 1×10^{-6}, 4×10^{-7}}. This spectrum of eigenvalues is quite interesting since three eigenvalues are extremely small, one is relatively small and the other one is large in magnitude. This set of closed-loop eigenvalues was used as the desired set of eigenvalues in designing controllers by eigenvalue assignment techniques.

In Table 4.8, feedback gain matrices derived from optimal control theory and two eigenvalue assignment techniques are shown along with the closed-loop eigenvalues and the design configurations that were employed. The controllers derived in Runs 16 and 17 are those with the smallest gain magnitudes that were obtained by trial and error; the largest gain elements are significantly smaller than the comparable elements in the optimal controller. A pair of complex eigenvalues was obtained in both Run 16 and Run 17. This creation of a pair of complex eigenvalues always occured when more than two open-loop eigenvalues were to be moved to very small values (~0). During this study, it was observed that the closed-loop eigenvalues were very sensitive to the numerical accuracy of the left eigenvector used at each recursive stage. This is especially true when some of the desired eigenvalues are near the origin and the word length of the digital computer is limited. For this reason, the creation of the complex eigenvalue pair could not be eliminated even by using "extended precision" on the IBM 360 computer.

Figures (4.12)-(4.14) compare the closed-loop responses for the optimal controller and controllers derived using two eigenvalue assignment techniques for step disturbances in feed flow rate, feed concentration and feed temperature. The optimal controller effectively

TABLE 4.8

Controllers Derived from Eigenvalue Assignment Techniques and Optimal Control Theory

Run No.	Design Configuration	$\underline{\underline{K}}_{FB}$					Calculated ρ_i
Optimal Controller	$J = \sum_{k=1}^{\infty} [\underline{x}(k)^T \underline{\underline{Q}}\underline{x}(k) + \underline{u}(k)^T \underline{\underline{R}}\underline{u}(k)]$	10.79	-1.61	-4.82	0.00	-19.58	1×10^{-6}
	$\underline{\underline{Q}}$ = diag [10, 1, 1, 10, 100]	5.35	0.36	0.54	0.00	12.50	4×10^{-7}
	$\underline{\underline{R}} = \underline{\underline{0}}$	7.52	1.27	0.18	24.62	32.70	5×10^{-6}
	Minimization of J						0.2706
							0.9002

Run No	λ_5	λ_4	λ_1	λ_3	λ_2						
						10.48	-6.83	-5.20	0.04	-16.79	$4\times10^{-6} \pm 6\times10^{-4} i$
16	1×10^{-6}	4×10^{-7}	5×10^{-6}	0.2706	0.9002	8.94	13.69	0.83	0.05	3.98	-8×10^{-4}
	Simultaneous Use of Controls					14.23	14.81	-0.46	24.68	24.97	0.2706
											0.9002

Table 4.8 - continued

Run No.	Design Configuration					$\underline{\underline{K}}_{FB}$	Calculated ρ_i
	λ_5	λ_4	λ_1	λ_3	λ_2	$$\begin{bmatrix} 2.97 & -3.53 & -4.30 & 0.00 & -16.85 \\ 12.92 & 2.56 & -0.42 & 0.00 & 2.60 \\ 24.62 & 22.74 & -0.17 & 24.62 & 23.11 \end{bmatrix}$$	$3\times10^{-6} \pm 2\times10^{-4} i$
17	1×10^{-6}	4×10^{-7}	5×10^{-6}	0.2706	0.9002		4×10^{-7}
	u_1	u_3	u_2	u_2	u_1		0.2706
	Separate Use of Controls						0.9002

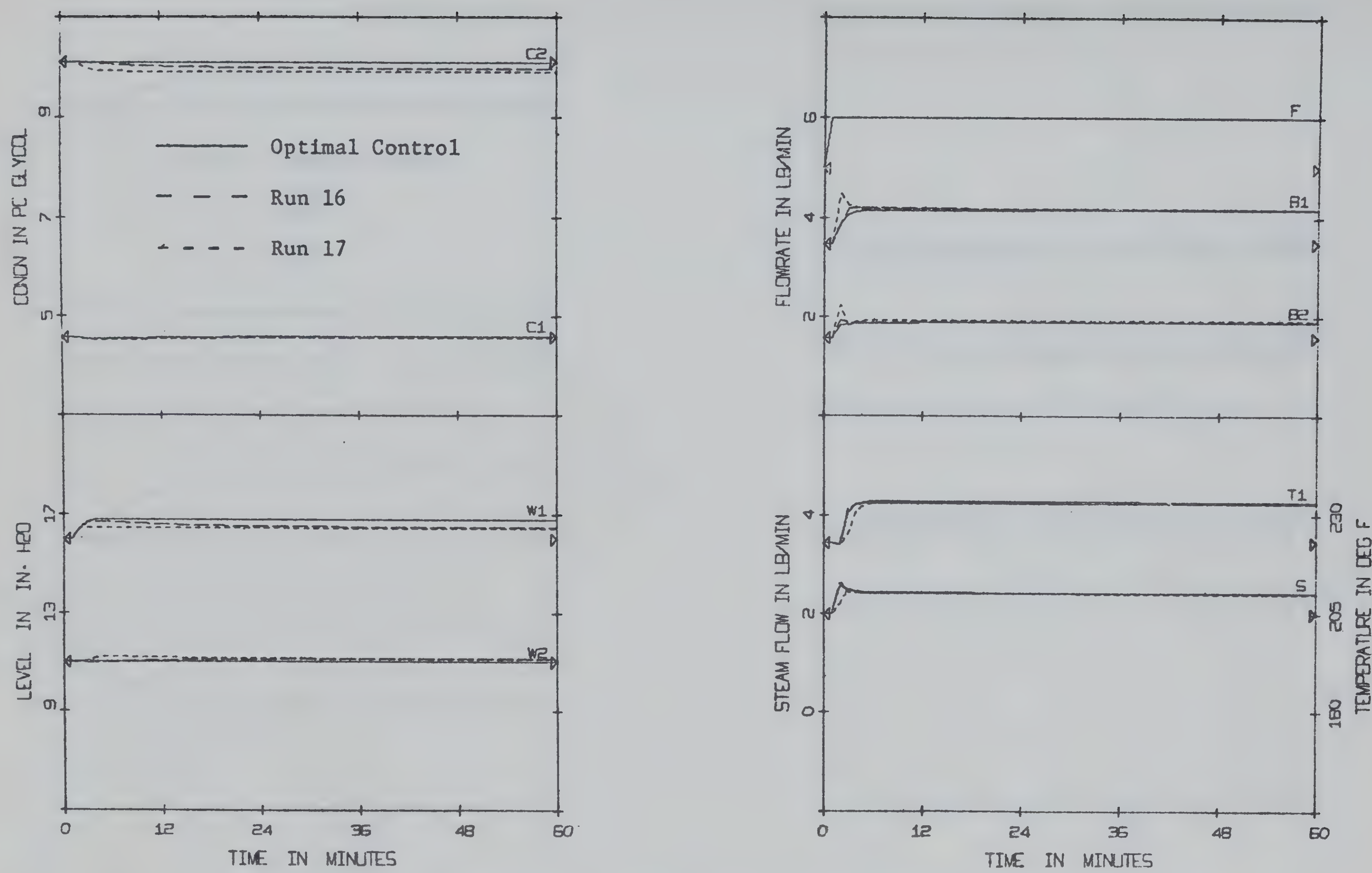

Figure 4.12 Comparison of an Optimal Controller with Controllers Derived from Eigenvalue Assignment (+20% F/P)

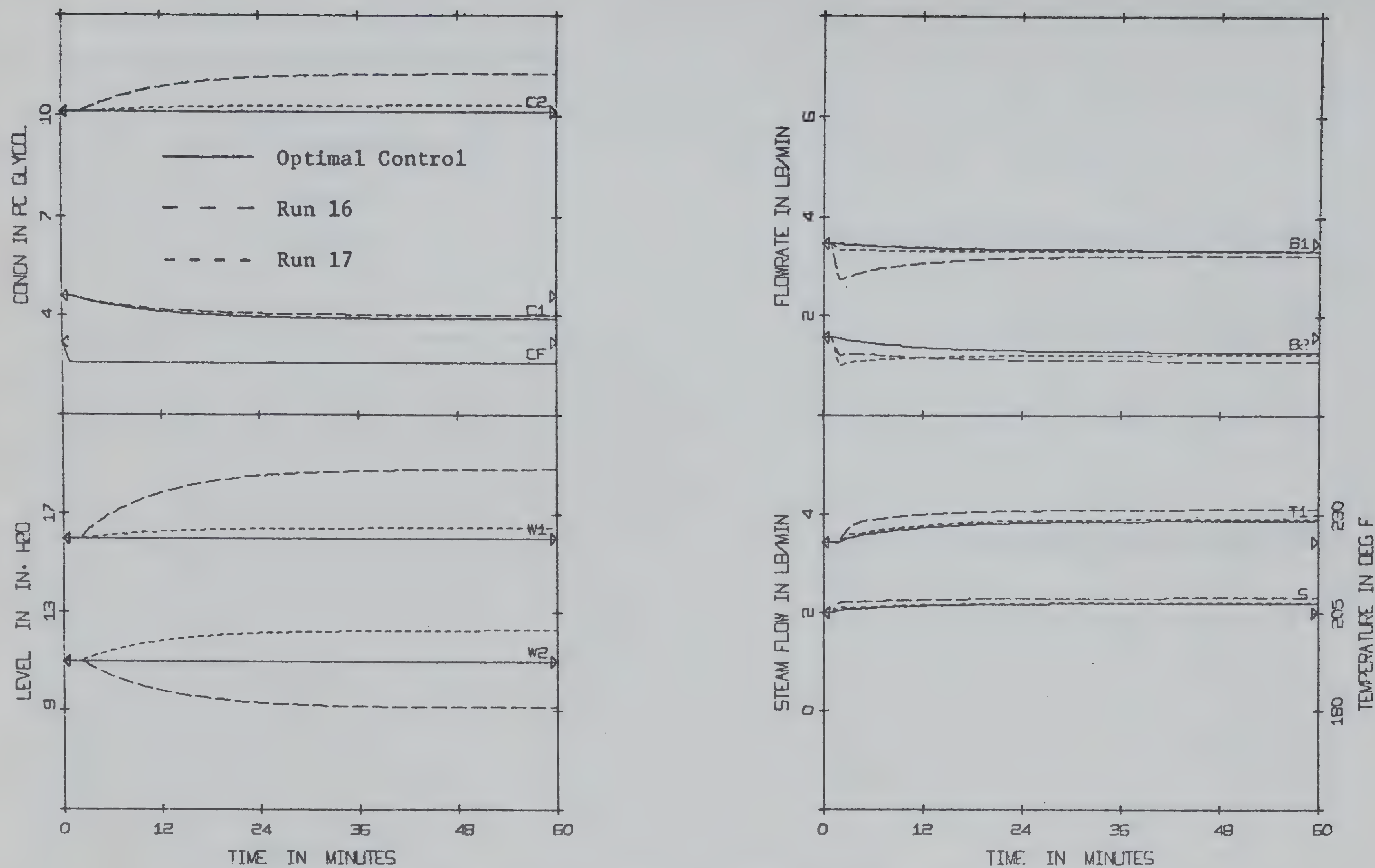

Figure 4.13 Comparison of an Optimal Controller with Controllers Derived from Eigenvalue Assignment (-20% CF/P)

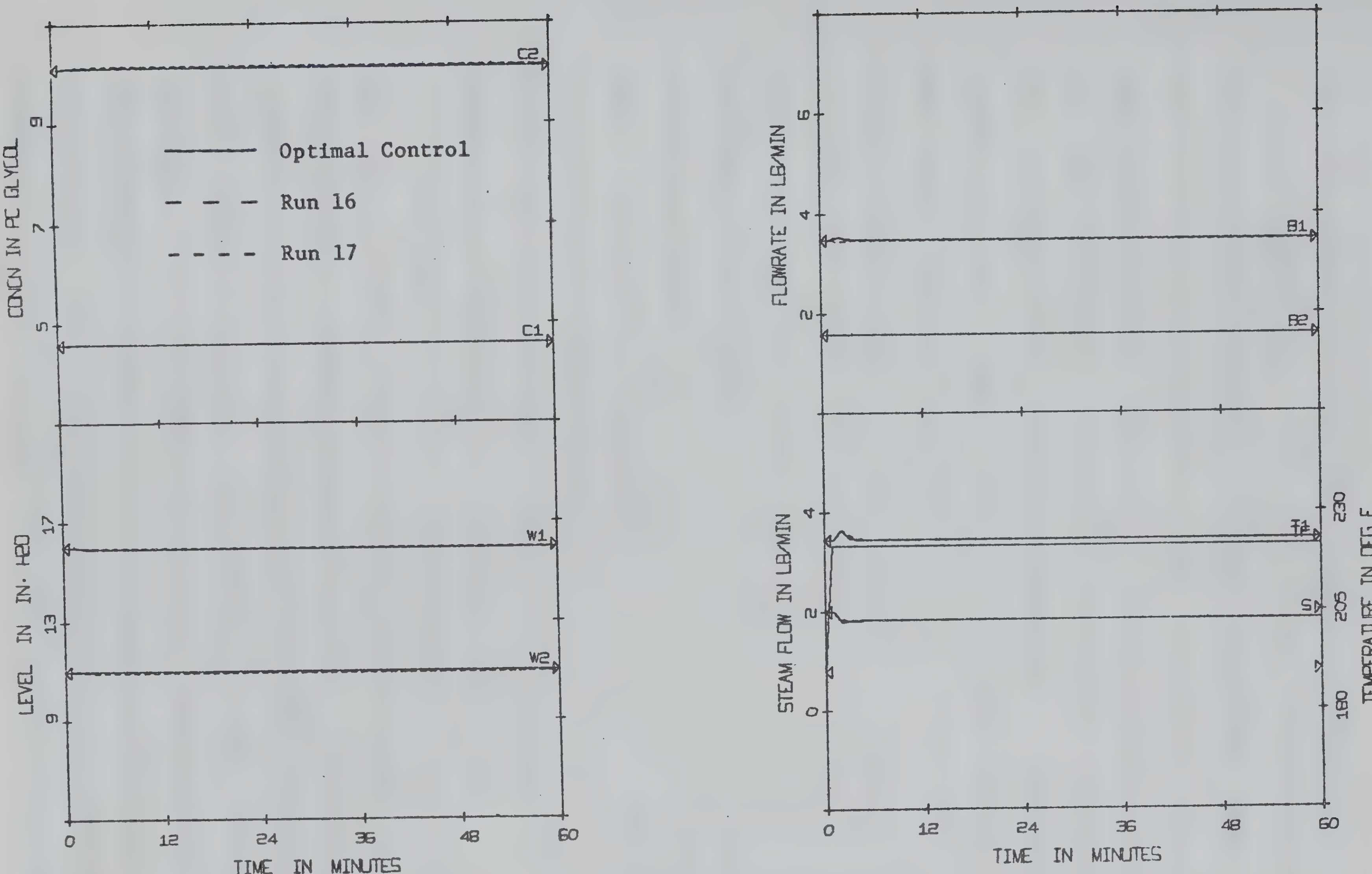

Figure 4.14 Comparison of an Optimal Controller with Controllers Derived from Eigenvalue Assignment (+20% TF/P)

reduces the influence of three different types of disturbances on the heavily weighted states while the performance of the other two controllers is still dependent on the type of disturbance present. Since the closed-loop eigenvalues are almost identical for these three closed-loop systems, the different dynamic responses are entirely due to the orientation of the closed-loop eigenvectors. However, direct comparison of the closed-loop eigenstructure is impossible because of the pair of complex closed-loop eigenvalues resulting from Runs 16 and 17. Instead, the eigenstructure of the closed-loop system obtained from Run 10 is compared with that of optimal controller to see the effect of the closed-loop eigenvector orientation. Although, the resulting closed-loop eigenvalues from Run 10 and the optimal controller are quite different individually, the magnitudes of gain elements are comparable and the closed-loop dynamics obtained from Runs 10, 16 and 17 show the same disturbance dependence. Furthermore, it will be seen that the size of the closed-loop eigenvalues is less important than the orientation of the closed-loop eigenvectors in determining the dynamic behavior of the closed-loop system.

In Tables 4.9 and 4.10, the eigenstructure of the closed-loop systems resulting from Run 10 and the optimal controller are presented along with the mode disturbance matrices. The eigenvectors in Tables 4.9 and 4.10 are normalized such that Equation (2-7) holds and the Euclidean norm of each left eigenvector is unity. Then the determinant of the left eigenvector matrix is an indication of how well distributed the eigenvectors of the closed-loop systems are in the state space. The effect of poorly distributed left eigenvectors is directly reflected in the magnitude of the elements in the right

TABLE 4.9

Eigenstructure and Mode Disturbance Matrix of the Closed-Loop System of Run 10

Eigenvalues, $\{\rho_i\}$:

$$\{\rho_1, \rho_2, \ldots, \rho_5\} = \{0.5,\ 0.4,\ 0.3,\ 0.2,\ 0.1\}$$

Left Eigenvector Matrix, $\underline{\underline{V}}$:

$$[\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_5] = \begin{bmatrix} 0.2190 & 0.1936 & 0.1582 & 0.1035 & -0.2005 \\ 0.7253 & 0.6994 & 0.6702 & 0.6264 & 0.2622 \\ -0.0110 & -0.0118 & -0.0110 & -0.0061 & 0.0435 \\ -0.0827 & -0.1097 & -0.1449 & -0.2014 & -0.5637 \\ -0.6473 & -0.6790 & -0.7105 & -0.7458 & -0.7559 \end{bmatrix}$$

Determinant, $|\underline{\underline{V}}|$ -4.2×10^{-8}

Right Eigenvector Matrix, $\underline{\underline{W}}$:

$$[\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_5] = \begin{bmatrix} -158.8 & 382.1 & -227.0 & 9.4 & 5.4 \\ 177.4 & -451.8 & 386.0 & -113.7 & 3.3 \\ -818.3 & 2800.4 & -3122.1 & 1179.2 & -44.1 \\ -153.3 & 350.2 & -638.2 & 275.5 & -16.9 \\ 173.6 & -492.3 & 490.4 & -179.5 & 8.5 \end{bmatrix}$$

Table 4.9 - continued

Mode Disturbance Matrix, $\underline{\underline{\theta}}^T \underline{\underline{V}}$:

$$[\underline{\underline{\theta}}^T \underline{v}_1, \underline{\underline{\theta}}^T \underline{v}_2, \ldots, \underline{\underline{\theta}}^T \underline{v}_5] = \begin{bmatrix} 0.0017 & -0.0004 & -0.0035 & -0.0085 & -0.0327 \\ 0.0559 & 0.0538 & 0.0514 & 0.0479 & 0.0194 \\ -0.0015 & -0.0016 & -0.0015 & -0.0010 & 0.0041 \end{bmatrix}$$

TABLE 4.10

Eigenstructure and Mode Disturbance Matrix of the Closed-Loop System from the Optimal Controller

Eigenvalues, $\{\rho_i\}$:

$$\{\rho_1, \rho_2, \ldots, \rho_5\} = \{0.9002,\ 0.2706,\ 5 \times 10^{-6},\ 1 \times 10^{-6},\ 4 \times 10^{-7}\}$$

Left Eigenvector Matrix, $\underline{\underline{V}}$:

$$[\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n] = \begin{bmatrix} 0.2910 & -0.4806 & -0.1905 & -0.3158 & 0.3540 \\ 0.7978 & 0.0252 & 0.0232 & 0.0074 & -0.0127 \\ -0.0048 & 0.0876 & 0.0953 & 0.0342 & -0.0558 \\ 0.0 & 0.0 & -0.0125 & -0.1302 & -0.6271 \\ -0.5280 & 0.8722 & 0.9767 & -0.9392 & 0.6914 \end{bmatrix}$$

Determinant, $|\underline{\underline{V}}|$ -2.3×10^{-2}

Right Eigenvector Matrix, $\underline{\underline{W}}$:

$$[\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_5] = \begin{bmatrix} 0.0161 & -3.305 & 3.003 & 0.0 & -0.0598 \\ 1.244 & 1.544 & -1.156 & -0.3904 & 0.1040 \\ -0.2296 & 13.47 & 19.47 & 6.578 & -1.753 \\ 0.0 & 0.0 & -0.3091 & -1.293 & -1.320 \\ -0.0040 & 0.6337 & -0.2679 & -0.6494 & 0.1401 \end{bmatrix}$$

Table 4.10 - continued

Mode Disturbance Matrix, $\underline{\underline{\theta}}^T \underline{\underline{V}}$:

$$[\underline{\underline{\theta}}^T \underline{v}_1, \underline{\underline{\theta}}^T \underline{v}_2, \ldots, \underline{\underline{\theta}}^T \underline{v}_5] = \begin{bmatrix} 0.0074 & -0.0605 & -0.0265 & -0.0364 & 0.0409 \\ 0.0617 & 0.0034 & 0.0034 & -0.0009 & 0.0002 \\ -0.0009 & 0.0134 & 0.0131 & -0.0008 & 0.0021 \end{bmatrix}$$

eigenvector matrix. The comparison of the numerical values of these determinants of the closed-loop systems derived from Run 10 and the optimal controller with the value of 0.45 for the open-loop system confirms the magnitude of elements shown in the respective right eigenvector matrix.

If a zero initial state, e.g., $\underline{x}(0) = \underline{0}$ and a constant disturbance vector, e.g., $\underline{d}(k) = \underline{d}$ is assumed, the state at the k^{th} sampling time is related to the eigenstructure of the closed-loop system (with distinct eigenvalues) in the following manner:

$$\underline{x}(kT) = \underline{\underline{W}}\underline{\underline{\Lambda}}(k)\underline{\underline{V}}^T\underline{\underline{\theta}}\underline{d}$$

or

$$\underline{x}(kT) = \sum_{i=1}^{n} \langle\underline{\underline{\theta}}^T\underline{v}_i,\underline{d}\rangle\rho_i(k)\underline{w}_i \tag{4-4}$$

where

$$\underline{\underline{\Lambda}}(k) = \text{diag}\ [\rho_1(k),\rho_2(k),\ldots,\rho_n(k)]$$

and

$$\rho_i(k) = \frac{1-\rho_i^k}{1-\rho_i} \qquad (i = 1,2,\ldots,n) \quad .$$

Equation (4-4) shows that the i^{th} column of the mode disturbance matrix $\underline{\underline{\theta}}^T\underline{\underline{V}}$ represents the influence of the j^{th} $(j = 1,2,3)$ disturbance on the states due to the orientation of the i^{th} left eigenvector relative to the j^{th} column vector of matrix $\underline{\underline{\theta}}$. Hence, if all the left eigenvectors happen to be oriented toward a direction

which has small components along some of the column vectors of the disturbance matrix $\underline{\underline{\theta}}$, then the effect of corresponding disturbances on the states will be insignificant. But, for other disturbances, the response of the closed-loop system will suffer from the large elements in the right eigenvector matrix. For the closed-loop system of Run 10, Table 4.9 shows that the first four left eigenvectors are clustered around a direction which has a significant component along the direction of the second column vector of matrix $\underline{\underline{\theta}}$. Only the last left eigenvector is fairly well separated from the other left eigenvectors. Consequently, the elements of the last right eigenvector are very small compared with those of the other right eigenvectors. This explains why the closed-loop response of Run 10 is poor for the feed concentration disturbance despite the small eigenvalues.

On the other hand, the left eigenvectors of the closed-loop system resulting from the optimal control law are fairly well distributed and prevent excessively large elements in the right eigenvector matrix. Considering the comparable magnitude of the elements in matrix $\underline{\underline{\theta}}^T\underline{\underline{V}}$ for the cases of Run 10 and optimal controller, the difference in the closed-loop dynamics is mainly to the directional effect of the right eigenvectors in the state space. Furthermore, although the optimal controller was obtained without explicit reference to the closed-loop system, the resulting closed-loop eigenstructure is such that the effects of the disturbances on the heavily weighted states are minimal. This is achieved not only be the small closed-loop eigenvalues but by eigenvector directions such that the combined effect of eigenvalues and the associated eigenvectors rejects the influence of the disturbances or non-zero initial state as much as

possible. However, even the optimal controller seems to be unable to regulate the heavily weighted states to the same extent depending on the disturbance. In Figure 4.12, despite the same weighting on the first and second effect holdups, it is seen that the response of the first effect level has a much larger offset than that of the second effect level. On the other hand, the first effect concentration, which is ten times less weighted than the level, exhibits smaller offset than the level. This leads to the conclusion that the direction of the closed-loop eigenvectors cannot be controlled arbitrarily, even if the closed-loop eigenvalues are not prespecified, when the system is to be stabilized and the number of controls is less than the number of states.

The comparison of the optimally controlled system with the closed-loop systems derived by the eigenvalue assignment techniques reveals the importance of eigenvector directions of the closed-loop system in the regulation of the states. It was shown that a closed-loop system with well distributed eigenvectors in the state space exhibits better dynamic performance than a closed-loop system with poorly distributed eigenvectors. Unless the number of the controls is equal to the number of states, there seems to be a limitation in controlling closed-loop eigenvectors when the system is to be stabilized.

4.5 PROPORTIONAL PLUS INTEGRAL CONTROL OF THE EVAPORATOR

The simulation studies of the evaporator presented in Section 4.4 confirmed the well-known fact that, in the presence of sustained disturbances, it is impossible to eliminate offsets

completely in any of the states using only a proportional controller. Hence, it is the purpose of this section to eliminate offsets in some of the states by using a PI controller designed by the eigenvalue assignment techniques presented in Chapter Three.

The state variables chosen for the integral control action and the resulting augmented open-loop system were presented in Section 4.2. This section is concerned with the effects of the following design factors:

i) the effects of the two algorithms used in the design of integral portion of the PI controllers,

ii) the effects of the design configurations used in the first step of the PI controller design, and

iii) the effects of the different sets of the closed-loop eigenvalues assigned in the first step of the PI controller design,

on the performance of the resulting systems.

As a reference controller for comparing the performance of the resulting closed-loop systems, an optimal PI controller was designed using the augmented open-loop system. Due to the increased dimension of the augmented system, a state weighting matrix, $\hat{\underline{\underline{Q}}}$, of dimension (n+r) was specified in the performance index. Furthermore, to reduce the magnitude of the gain elements, a non-zero control, weighting matrix, $\underline{\underline{R}}$, was selected. The numerical values of the matrices, $\hat{\underline{\underline{Q}}}$ and $\underline{\underline{R}}$, used in the design of an optimal controller are given by;

$$\hat{\underline{\underline{Q}}} = \text{diag}\,[10,\ 1,\ 1,\ 10,\ 100,\ 1,\ 1,\ 1]$$

$$\underline{\underline{R}} = \text{diag}\,[0.05,\ \ 0.05,\ \ 0.05]\ .$$

The resulting optimal PI controller is given in Table 4.12. The corresponding closed-loop eigenvalues are,

$$\{\rho_i\} = \{0.1299,\ 0.3366\pm 0.1249i,\ 0.6566,\ 0.7549,\ 0.7824,\ 0.8980,\ 0.9000\}\ .$$

In the design of the PI controllers using eigenvalue assignment techniques, these eigenvalues were specified as the desired closed-loop eigenvalues. However, since the optimal PI controller produced a pair of complex conjugate eigenvalues, the desired eigenvalue corresponding to these complex eigenvalues were replaced by two real eigenvalues. This was done to avoid the difficulty of handling complex eigenvectors.

The design configurations used in the eigenvalue assignment techniques are summarized in Table 4.11. The corresponding PI controllers are tabulated in Table 4.12.

4.5.1 Comparison of the Recursive and Simultaneous Designs

The design configurations used in Runs 18 and 19 differ only in the second step, where the three unity eigenvalues are shifted to the desired locations using recursive and simultaneous approaches, respectively. Controls were used simultaneously with the $\underline{g}_i$ vectors calculated from Equation (2-32) at each stage of the first design step. The design configurations of Runs 20 and 21 are analogous to those of Runs 18 and 19, except that, in the first step, controls were used separately to shift the five eigenvalues of matrix $\underline{\phi}(T)$. The pairing of controls and eigenvalues that were used in Runs 20 and 21 are shown in Table 4.11.

Although the same design configurations were used in the

TABLE 4.11

Design Configurations Used in PI Controller Design

Run No.	Configuration Used in Designing $\underline{\underline{K}}^{(*)}$					Configuration Used in Designing $\underline{\underline{K}}_I^{(**)}$		
	λ_5	λ_4	λ_1	λ_3	λ_2	λ_6 (1.0)	λ_7 (1.0)	λ_8 (1.0)
18	0.1299	0.2116	0.6566	0.4615	0.9000	0.7549	0.7824	0.8981
	Simultaneous Use of Controls					Recursive Design		
	λ_5	λ_4	λ_1	λ_1	λ_2	λ_6 (1.0)	λ_7 (1.0)	λ_8 (1.0)
19	0.1299	0.2116	0.6566	0.4615	0.9000	0.7549	0.7824	0.8981
	Simultaneous Use of Controls					Simultaneous Design		
	λ_5	λ_4	λ_1	λ_3	λ_2	λ_6 (1.0)	λ_7 (1.0)	λ_8 (1.0)
20	0.1299	0.2116	0.6566	0.4615	0.9000	0.7549	0.7824	0.8981
	u_1	u_3	u_2	u_2	u_1	Recursive Design		

Table 4.11 - continued

Run No.	Configuration Used in Designing $\underline{\underline{K}}^{(*)}$					Configuration Used in Designing $\underline{\underline{K}}_I^{(**)}$		
	λ_5	λ_4	λ_1	λ_3	λ_2	λ_6 (1.0)	λ_7 (1.0)	λ_8 (1.0)
21	0.1299	0.2116	0.6566	0.4615	0.9000	0.7549	0.7824	0.8981
	u_1	u_3	u_2	u_2	u_1	Sinultaneous Design		
	λ_5	λ_4	λ_1	λ_3	λ_2	λ_6 (1.0)	λ_7 (1.0)	λ_8 (1.0)
22	0.5	0.4	0.2	unchanged	0.6	0.7549	0.7824	0.8981
	Simultaneous Use of Controls					Simultaneous Design		

(*) The first row indicates the sequence of changing original system eigenvalues and the second and third rows indicate the corresponding closed-loop eigenvalues and controls.

(**) In the Recursive Design, the first and second rows have the same meaning as in the $\underline{\underline{K}}$ design. But, in the Simultaneous Design, they only indicate the repeated unity eigenvalue and the desired closed-loop eigenvalues.

TABLE 4.12

Feedback Controller Matrices

Run No.	Proportional Controller, $\underline{\underline{K}}_{FB}$	Integral Controller, $\underline{\underline{K}}_{I}$
Optimal Controller	$\begin{bmatrix} 8.21 & -1.24 & -3.64 & 0.14 & -15.45 \\ 4.54 & 0.37 & 0.55 & -1.28 & 9.07 \\ 4.24 & 1.17 & -0.06 & 12.25 & 14.22 \end{bmatrix}$	$\begin{bmatrix} 1.27 & 0.03 & -1.43 \\ 0.79 & -0.30 & 0.89 \\ 0.65 & 1.94 & 1.31 \end{bmatrix}$
18	$\begin{bmatrix} 13.11 & 10.00 & -3.74 & 3.30 & -15.86 \\ 2.72 & -7.43 & 1.16 & 4.13 & 18.92 \\ 5.36 & -10.35 & -1.32 & 16.42 & 23.26 \end{bmatrix}$	$\begin{bmatrix} -0.13 & -0.89 & -2.55 \\ 1.97 & 1.21 & 1.65 \\ 1.97 & 2.72 & 3.16 \end{bmatrix}$
19	$\begin{bmatrix} 7.87 & -7.21 & -3.81 & 3.57 & -5.36 \\ 3.53 & -5.67 & 1.16 & 3.81 & 16.74 \\ 10.35 & 0.48 & -1.73 & 15.72 & 11.85 \end{bmatrix}$	$\begin{bmatrix} 1.96 & 1.06 & -0.43 \\ 2.01 & 1.11 & 1.21 \\ 2.45 & 2.85 & 1.05 \end{bmatrix}$

Table 4.12 - continued

Run No.	Proportional Controller, $\underline{\underline{K}}_{FB}$	Integral Controller, $\underline{\underline{K}}_{I}$
20	$\begin{bmatrix} 7.92 & 4.67 & -4.07 & -0.20 & -20.53 \\ 0.02 & -12.87 & -0.04 & 0.36 & 11.32 \\ 10.34 & -4.89 & -0.11 & 20.13 & 32.11 \end{bmatrix}$	$\begin{bmatrix} -0.11 & -0.68 & -2.42 \\ 2.22 & 1.26 & 1.86 \\ 2.22 & 2.54 & 3.14 \end{bmatrix}$
21	$\begin{bmatrix} 5.56 & -1.89 & -3.99 & 0.00 & -16.27 \\ 4.18 & -0.56 & -0.07 & 0.00 & 3.22 \\ 10.27 & -13.57 & -0.51 & 20.54 & 35.00 \end{bmatrix}$	$\begin{bmatrix} 0.61 & 0.00 & -1.49 \\ 0.75 & 0.00 & 0.16 \\ 4.46 & 3.96 & 2.21 \end{bmatrix}$
22	$\begin{bmatrix} 5.37 & -22.62 & -3.43 & 5.03 & 8.17 \\ 6.58 & -6.28 & 0.31 & 3.33 & 16.67 \\ 1.84 & -3.87 & -0.04 & 15.71 & 20.53 \end{bmatrix}$	$\begin{bmatrix} 4.45 & 1.47 & -0.65 \\ 1.20 & 0.79 & 1.77 \\ -0.50 & 2.57 & 1.84 \end{bmatrix}$

first steps of Runs 18 and 19, significantly different proportional feedback gains were obtained. Similarily, the proportional feedback matrices resulting from Runs 20 and 21 are quite different. However, the general features of the integral feedback matrix, including the signs of the gain elements, seem to be preserved if the same design policy is used in the second step of the design. This can be seen from a comparison of the $\underline{\underline{K}}_I$ matrices resulting from Runs 18 and 20, and Runs 19 and 21.

The transient responses of the closed-loop systems using the PI controllers of Runs 18 and 19 are shown in Figures 4.15 to 4.17 for step disturbances of +20% in F, CF and TF, respectively. The transient response for the optimal PI controller was also plotted for purposes of comparison. As in the case of proportional control, a disturbance in feed concentration has the most significant effect and a disturbance in feed temperature has only a negligible effect on the dynamic behavior of the closed-loop system. In Figures 4.18 and 4.19, the closed-loop response using the PI controllers derived in Runs 20 and 21 were plotted for step disturbances in F and CF only, since the effect of the disturbance in TF was also negligible in these two runs.

The transient response varies depending on not only the disturbance but also on the design method used in the second step of PI controller design. The comparison of transient responses in Figures 4.15 and 4.19 indicates that the new steady states are reached faster when the simultaneous design is used in the second step of PI controller design than when the recursive design is used.

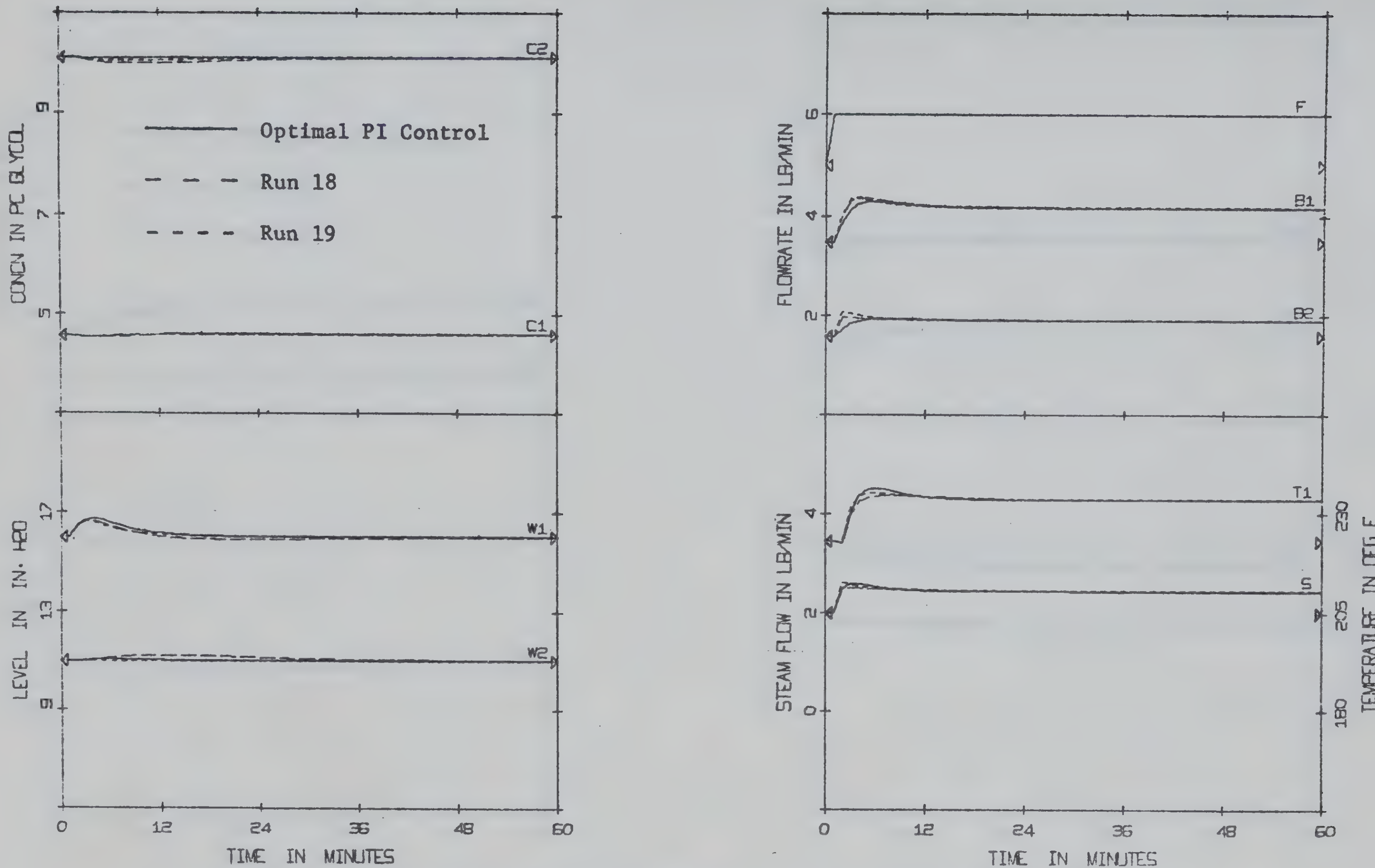

Figure 4.15 Simulated Effect of Different Integral Controller Design when Simultaneous Controls are Used (+20% F/PI)

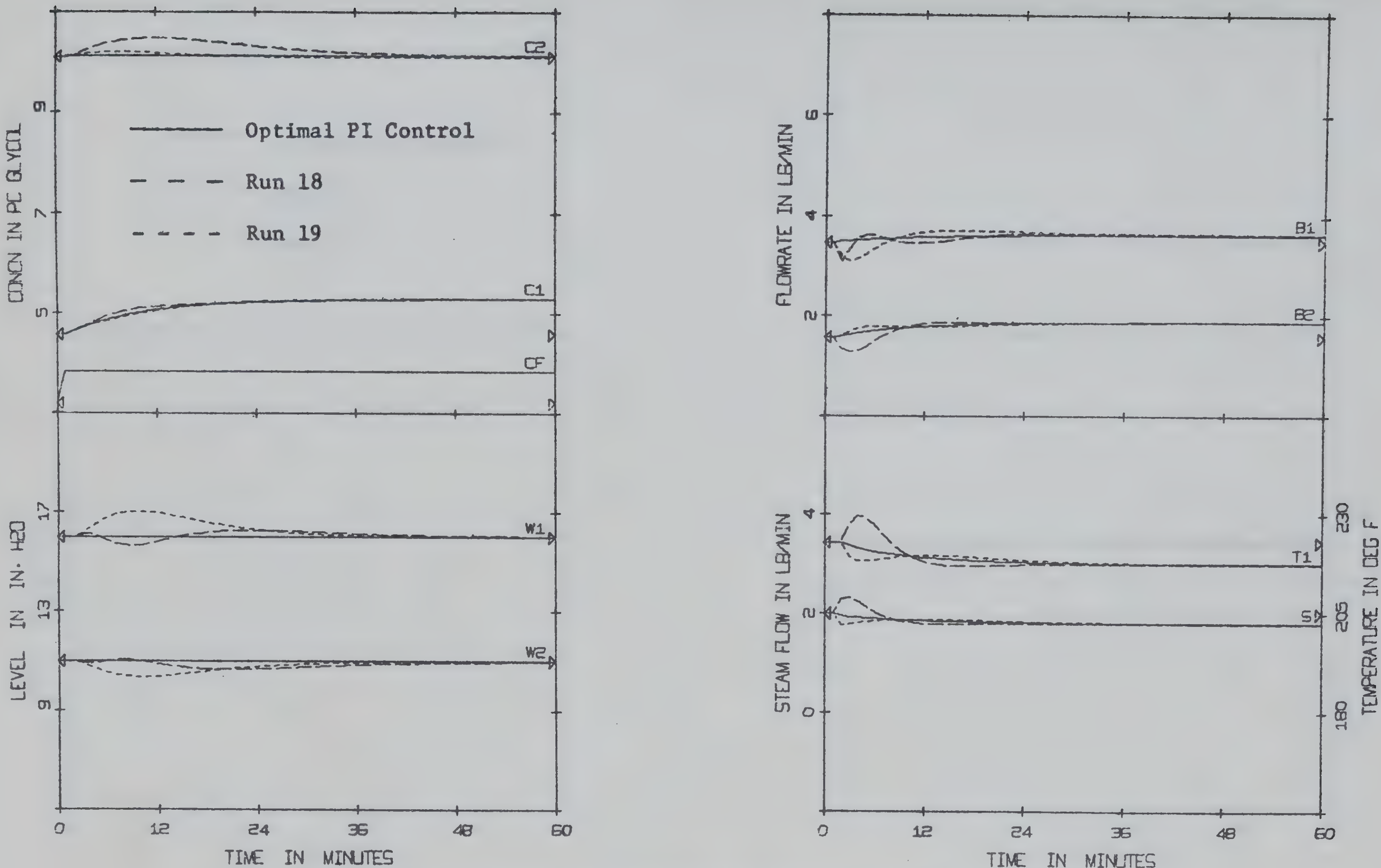

Figure 4.16 Simulated Effect of Different Integral Controller Designs when Simultaneous Controls are Used (+20% CF/PI)

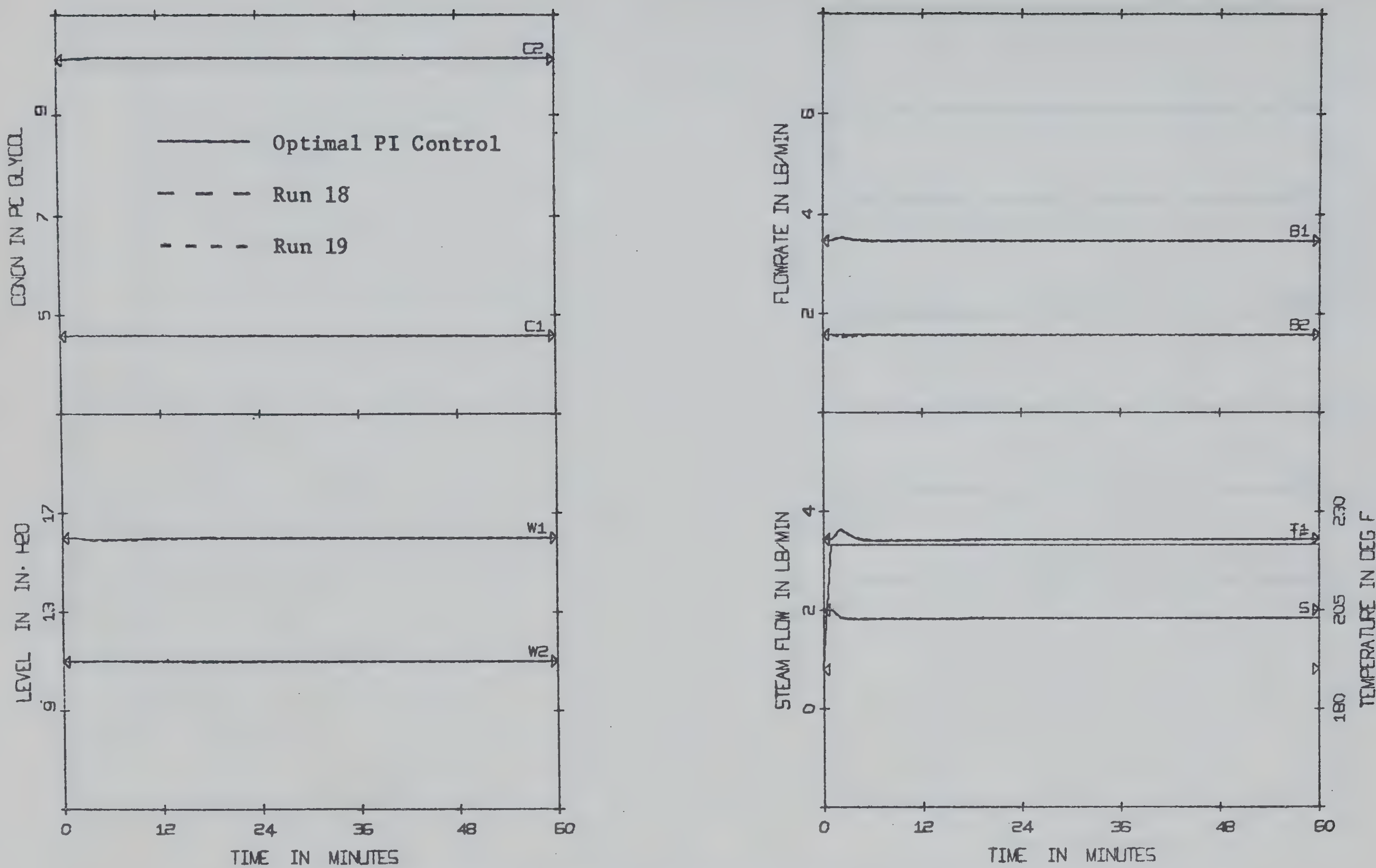

Figure 4.17 Simulated Effect of Different Integral Controller Designs when Simultaneous Controls are Used (+20% TF/PI)

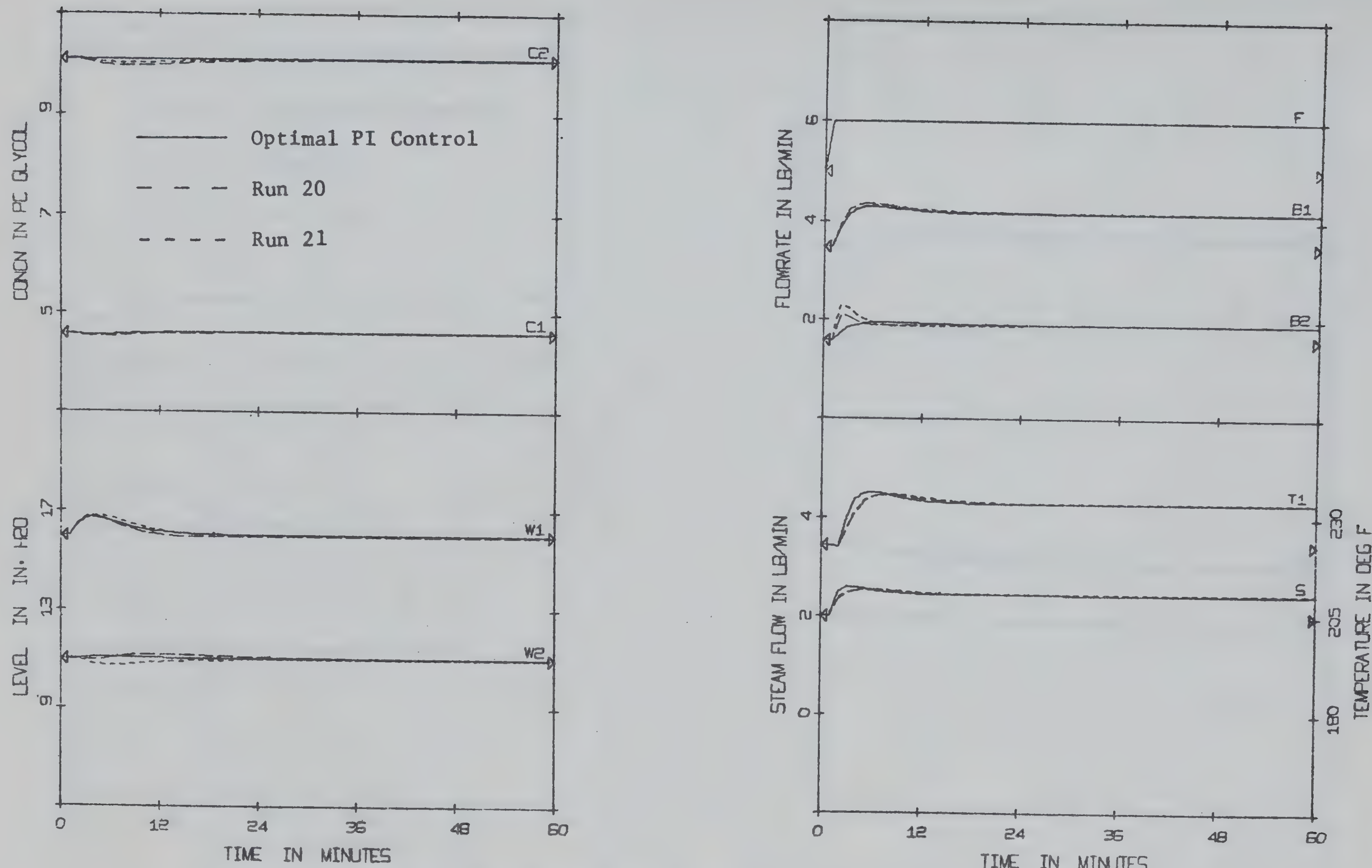

Figure 4.18 Simulated Effect of Different Integral Controller Designs when Separate Controls are Used (+20% F/PI)

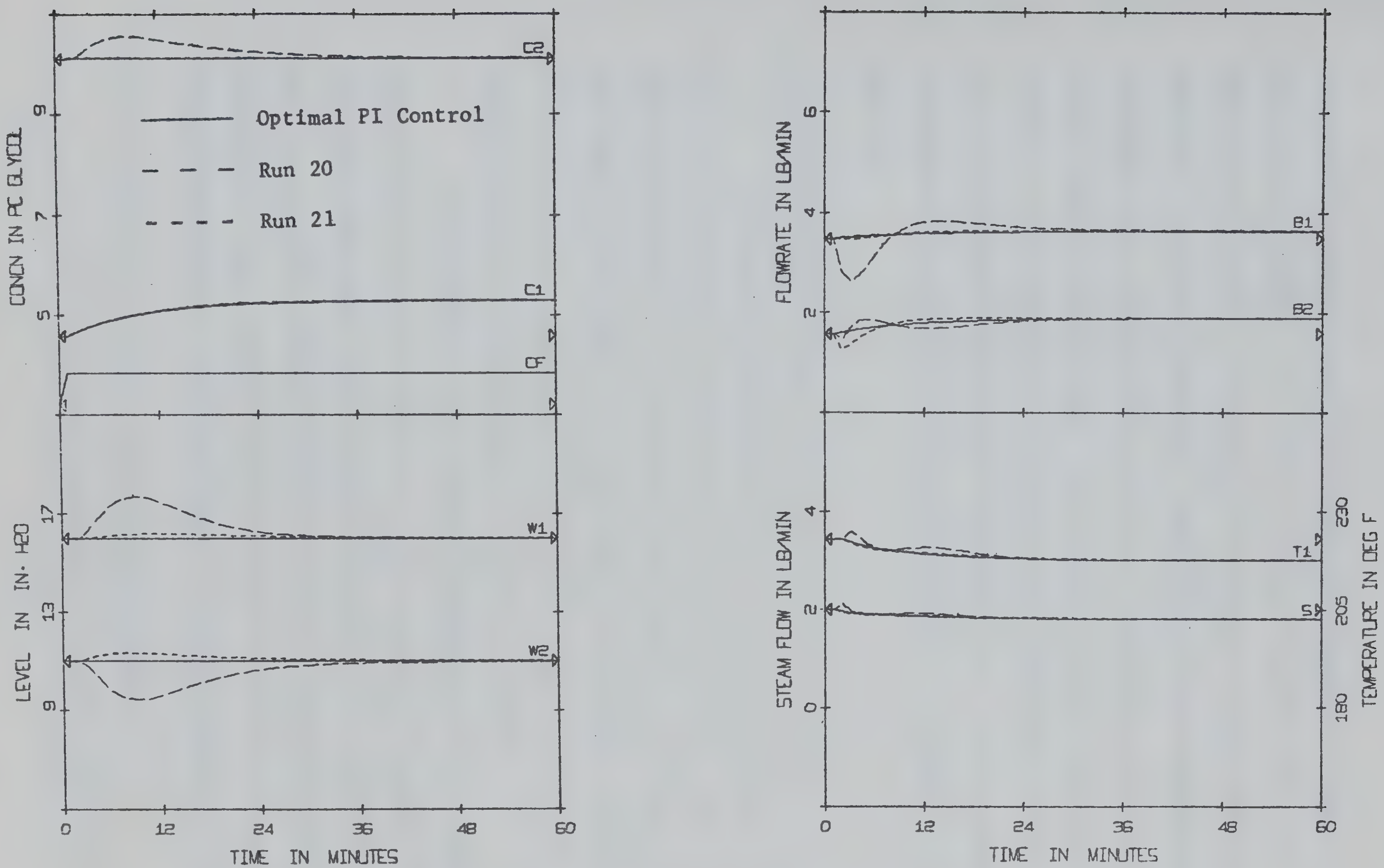

Figure 4.19 Simulated Effect of Different Integral Controller Designs when Separate Controls are used (+20% CF/PI)

This trend is apparent in all the responses regardless of which disturbance is present. For both the design methods, a feed flow disturbance produces the same type of transient response, but the response to the feed concentration disturbance changed significantly depending on the design method used in the second step. However, for the product concentration, C_2, which is the most important variable in this particular application, the simultaneous design approach provides better transient responses (i.e., smaller deviations from steady state) than the recursive approach (cf., Figures 4.16 and 4.19).

4.5.2 Effect of the Design Policy Used in the First Step

The design configurations of Runs 18 and 20 differ only in the first design step and similarily for Runs 19 and 21. Hence, the difference between the closed-loop systems reflects the effect of the design approach used in the first step of the design. However, it is difficult to generalize about the effects on the feedback control matrices and the closed-loop transient responses.

From Equation (3.56), the final expression for the proportional feedback matrix is given as, $\underline{\underline{K}}_{FB} = [\underline{\underline{K}} - T\underline{\underline{K}}_I \underline{\underline{T}}_r \{\underline{\underline{I}}_n + (\underline{\underline{\phi}} + \Delta\underline{\underline{K}} - \underline{\underline{I}}_n)^{-1}\}]$. Since this expression shows the explicit relationship between the $\underline{\underline{K}}$ and $\underline{\underline{K}}_I$ controller matrices designed in the first and second step, the structure of $\underline{\underline{K}}_{FB}$ and the transient response of the closed-loop system depend significantly on the magnitude of the gain elements of matrix $\underline{\underline{K}}_I$.

4.5.3 Effect of the Desired Closed-Loop Eigenvalues

In Run 22, only four eigenvalues of matrix $\underline{\underline{\phi}}(T)$ were

shifted to the locations given in Table 4.10. The other design options are the same as in Run 19.

A comparison of the resulting feedback matrices for Runs 19 and 22 does not reveal any common features between controllers. The structure of the controller matrices is quite different when a different set of eigenvalues are specified in the first step, even if the design configurations are identical in the second step.

The transient response of the closed-loop system of Run 22 is quite disturbance-dependent as was the case for other closed-loop systems. A disturbance in F results in a response similar to the one in Figure 4.15 and the response to the TF disturbance is very similar to Figure 4.17. For this reason, only the response to the CF disturbance is shown in Figure 4.20. It is apparent from Figure 4.20 that the closed-loop eigenvalues assigned in the first design step have a significant effect on the shape of the transient response of the resulting closed-loop system.

4.5.4 Summary and Interpretation

This summary is based on the performance of the closed-loop systems which were presented in this section.

1) As expected from theoretical considerations, offsets were eliminated in three states, namely W_1, W_2 and C_2, for all three constant disturbances. The elimination of offsets can also be shown analytically using the discrete representation of the augmented system but a rather lengthy derivation is required.

2) For a given disturbance, each of the five state variables converged to the same steady-state value regardless of which PI con-

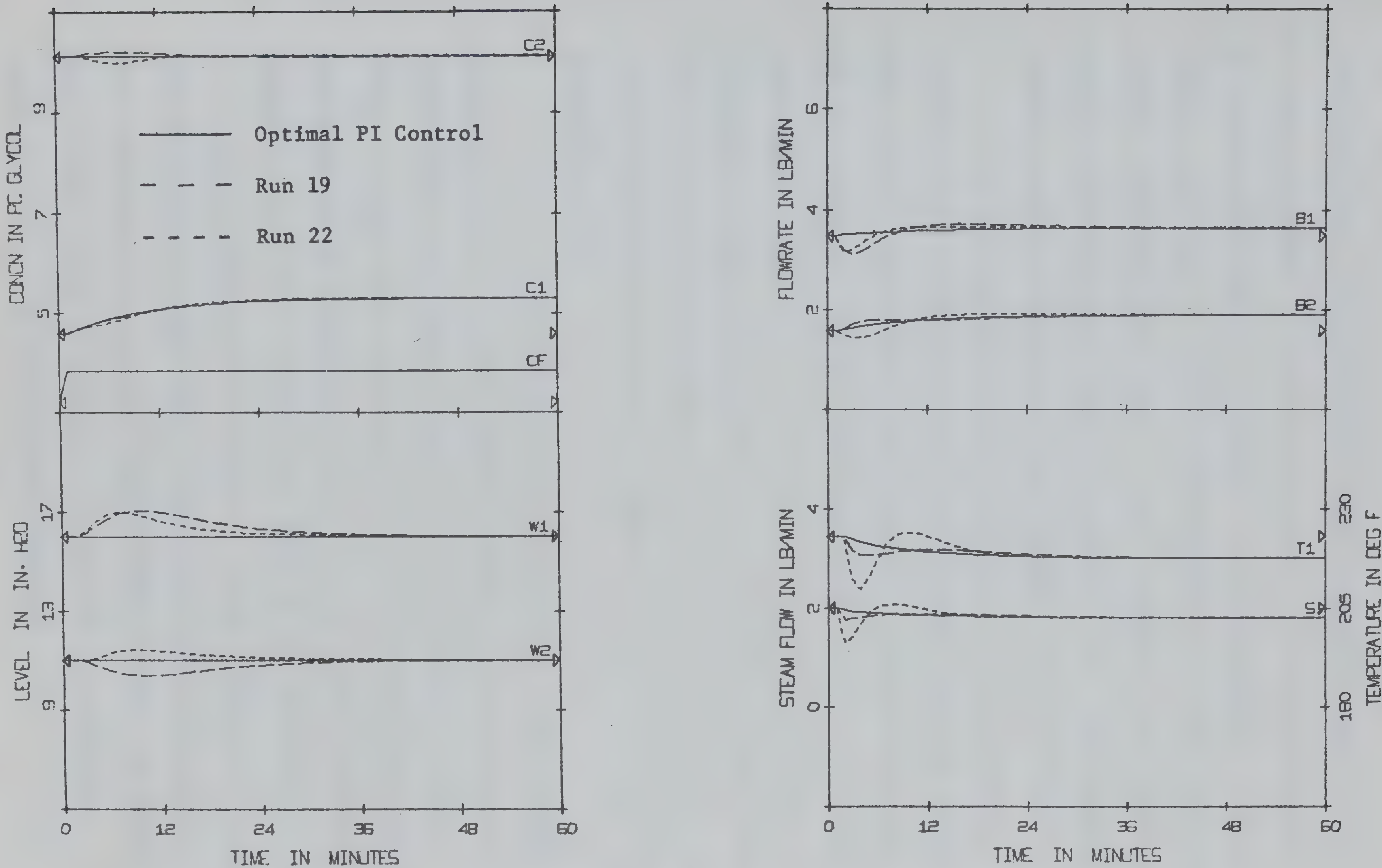

Figure 4.20 Simulated Effect of the Desired Closed-Loop Eigenvalues (+20% CF/PI)

troller was used. This phenomenon can be explained qualitatively in terms of degrees of freedom or in terms of the controllability of the augmented system. For example, the controllability condition indicates that only r $(\leq m)$ state variables can be used in integral control. This means that only r state variables can be independently driven to arbitrary values and the remaining $(n-r)$ state variables are then fixed since there are no more degrees of freedom available.

3) The transient responses of the closed-loop systems resulting from PI controllers are highly dependent upon the disturbance present. As observed in the study of proportional controls, the CF disturbance has the most significant effect on the closed-loop dynamics and the TF disturbance has a negligible effect. Since the recursive design method used in the first design step was identical to that used in the proportional controller design, this sensitivity to disturbances seems to be due to the recursive control policy used in the first step.

4) The simultaneous design method tends to give better transient responses, namely, smaller deviations from the desired steady states and a faster recovery from disturbances. In this study, a diagonal matrix consisting of the desired eigenvalues was used as matrix $\underline{\underline{D}}$ in Equation (3-55). Clearly, the eigenvectors of matrix $\underline{\underline{D}}$ are less colinear than those of the closed-loop $r \times r$ matrix designed by the recursive design. Thus, it is expected that the resulting eigenvectors of the augmented system are also well-distributed when the simultaneous design is used. In view of the discussion given in Section 4.4, this could be a possible explanation for the better transient responses.

CHAPTER FIVE

CONCLUSIONS

The purpose of this work was to investigate the application of existing eigenvalue assignment techniques based on state feedback to the design of multivariable control systems. In particular, an evaluation of the design options available when the eigenvalue assignment was based on a modal analysis, was one of the main objectives of this study. The other objective was to extend eigenvalue assignment techniques to the design of multivariable PI controllers.

A new theoretical result for PI control systems is that the integral control matrix must have full rank in order to arbitrarily assign all of the eigenvalues of the closed-loop system. The constructive proof of the proposition provided the basis for two modified alogrithms for PI controller design by eigenvalue assignment techniques.

Proportional controllers were designed for a double effect evaporator using several recursive approaches and the resulting closed-loop dynamics were simulated. In general, the simulation results revealed that the performance of the closed-loop evaporator system was highly disturbance-dependent and not always satisfactory when these proportional controllers were used. The unsatisfactory behavior of the closed-loop system was mainly due to the orientation of closed-loop eigenvectors in the state space. It was shown in simultation studies that the PI controllers designed by eigenvalue assignment techniques were able to eliminate offsets in the state variables

subjected to integral control.

5.1 SUMMARY

Since a rather detailed discussion of the simulation results for the evaporator system was given in each section of Chapter Four, only the general conclusions from the simulation studies are summarized here.

Conclusions from the Proportional Control Studies

The design options available in the recursive design methods have significant effects both on the dynamics of the closed-loop systems and the resulting controller matrices. In this regard, the sequence in which the open-loop eigenvalues are changed to the desired closed-loop eigenvalues had the most significant influence on the performance of the closed-loop system. However, effective utilization of these and the other design options was not obvious a priori and was generally achieved by trial and error.

For the same design configuration and desired closed-loop eigenvalues, the closed-loop dynamics became more satisfactory as the largest feedback gain element was reduced.

The new steady state was reached faster when the closed-loop eigenvalues were small as would be expected, but the performance of the closed-loop system did not necessarily improve as the magnitude of the closed-loop eigenvalues became smaller. Furthermore, changing all the controllable open-loop eigenvalues to smaller values did not necessarily result in an improved dynamic response of the closed-loop system in comparison with the case where only some of the con-

trollable open-loop eigenvalues were changed. Since the closed-loop dynamic behavior is determined by both eigenvalues and eigenvectors, manipulation of only eigenvalues is not always sufficient to improve the dynamic response of the closed-loop system.

The closed-loop dynamics were very sensitive to the type of disturbance present. The feed concentration disturbance had the most significant effect on the dynamics of all the closed-loop systems while a feed temperature disturbance had very little effect. The disturbance-dependence of the closed-loop systems was due to the orientation of the closed-loop eigenvectors in the state space. Furthermore, the closed-loop eigenvectors tended to be oriented in the same direction in the state space. This could not be directly controlled by the designer using eigenvalue assignment techniques.

In the evaporator simulation study, the closed-loop eigenvalues assigned at each recursive step were very sensitive to the numerical accuracy of the updated eigenvectors when the magnitude of the desired eigenvalues was small. Hence, in some applications, the straightforward algorithm for updating eigenvectors may not be accurate enough to assign the desired eigenvalues, unless the numerical techniques are carefully evaluated.

In conclusion, the simulation results revealed that in process control assigning eigenvalues of the closed-loop system is not sufficient and the closed-loop eigenvectors are also important to the same extent in ensuring satisfactory performance of the closed-loop system. The eigenvalue assignment techniques guarantee only the shift of open-loop eigenvalues to the desired locations but does not assign the closed-loop eigenvectors unless the available

design freedom can be exploited in some systematic way. However, the utilization of this design freedom will, in general, result in a non-linear optimization problem. Thus simplicity, which is the main advantage of eigenvalue assignment techniques, will be lost.

Conclusions from the Proportional Plus Integral Control Studies

The PI controllers designed by eigenvalue assignment techniques eliminated offsets in the properly chosen state variables of the evaporator system, regardless of the type of sustained disturbance. However, comparison of an optimal PI controller with PI controllers designed by eigenvalue assignment techniques showed that, although the same steady-state was reached, the transient response of the closed-loop system was more satisfactory when the optimal PI controller was used. Furthermore, for the PI controller derived from eigenvalue assignment techniques, the transient responses of the closed-loop systems depended on the types of disturbances present and the design configurations used. In view of the important role of the eigenvectors in the dynamic behavior of a system, this variation in the closed-loop dynamics was due to the eigenvector orientation, which depended on the design configurations used.

For the two algorithms presented in Chapter Three, all the closed-loop systems showed the same disturbance-dependence. Since this reflects the initial design policy used in the first step, a satisfactory proportional controller, $\underline{\underline{K}}$, is more important than the integral control matrix, $\underline{\underline{K}}_I$, if the two step design scheme is to be used. However, the simultaneous design approach used in the second step, in general, gave better transient responses of those

selected state variables than did the recursive approach. This again reflects the importance of the closed-loop eigenvectors in the dynamic behavior of the closed-loop system.

In conclusion, the simplicity of eigenvalue assignment techniques can be advantageously used in eliminating offsets in selected states. For the two step design schemes used in this study, more emphasis should be directed to the design of the initial proportional control matrix, $\underline{\underline{K}}$, to get satisfactory transient responses. However, if the standard eigenvalue assignment techniques are applied directly to the augmented system, the relation between eigenvectors and eigenvalues is expected to be more involved due to the increased dimension of the system.

5.2 FUTURE WORK

Although the eigenvalue assignment techniques are convenient from a computational viewpoint, there are serious disadvantages for process control applications. This is mainly due to the fact that no systematic way of specifying the closed-loop eigenvectors is available at the present time. The manipulation of available design freedoms in a trial and error manner is not always successful in a reasonable period of time and becomes more tedious as the number of states increase.

In this respect, future work should be directed to the establishment of desirable closed-loop eigenvector configurations in state space. Since the closed-loop eigenvectors depend on the closed-loop eigenvalues and the design freedom is mainly provided by an arbitrary set of vectors $\{\underline{g}_i\}$, the relationships among the

closed-loop eigenvectors, closed-loop eigenvalues and the set of vectors $\{\underline{g}_i\}$ should be determined for the successful application of eigenvalue assignment techniques to the process control problem.

Another importnat area where the future work is needed, is the minimization of gain elements in view of the possible limitations on the magnitude of available controls (i.e., control constraints). For multi-input systems, the gain elements are related to the selection of an arbitrary vector set, $\{\underline{g}_i\}$, and the relationship between the overall controller matrix and this set of vectors, $\{\underline{g}_i\}$ would be a valuable asset for the control system design.

These two unresolved problems in state feedback control are also important because the incomplete state or output feedback system design problem would profit from a fuller understanding of the state feedback problem.

However, the simulation results should be verified experimentally before proceeding with a further investigation of eigenvalue assignment techniques. Unfortunately, the experimental evaluation of the simulated results for the evaporator system had to be postponed due to the equipment difficulties. Hopefully, the experimental studies will be possible in the near future.

NOMENCLATURE

Alphabetic

$\underline{\underline{A}}$	system matrix of a continuous-time model
$\underline{b}$	control coefficient vector
$\underline{\underline{B}}$	control coefficient matrix
B_1	first effect bottoms flow
B_2	second effect bottoms flow
$\underline{\underline{C}}$	closed-loop system matrix
C_1	first effect concentration
C_2	product concentration
C_F	feed concentration
d_i	disturbance element
$\underline{d}$	disturbance vector
$\underline{\underline{D}}$	disturbance coefficient matrix or a desired closed-loop matrix defined in (3-30)
$f(\ )$	characteristic polynomial of a square matrix
$\underline{f}$	column vector defined in (2-3)
g_i	element of vector $\underline{g}$
$\underline{g}$	column vector defined in (2-3)
$\underline{\underline{G}}$	square matrix defined in (3-33)
F	feed flow rate
h_1	first effect enthalpy
h_F	feed enthalpy
$\underline{\underline{H}}$	mode controllability matrix defined in (2-10)
$\underline{\underline{H}}_1$	mode controllability matrix defined in (3-17)
$\underline{\underline{H}}_2$	mode controllability matrix defined in (3-18)

i	increment counter
$\underline{\underline{I}}$	identity matrix
$\underline{\underline{J}}$	Jordan canonical form of a square matrix
k	time counter
$\underline{\underline{K}}$	feedback control matrix or initial feedback control matrix defined in (3-11)
$\underline{\underline{K}}_{FB}$	proportional feedback control matrix
$\underline{\underline{K}}_{I}$	integral feedback control matrix
m	number of controls
n	number of states
$\underline{\underline{O}}$	null matrix
p	number of eigenvalues to be changed
q	number of disturbances
q_d	eigenvector updating coefficient
r	number of integrated states
S	steam flow rate
T	discretization interval
T_1	first effect temperature
T_F	feed temperature
$\underline{\underline{T}}_r$	coefficient matrix of integrated state vector
u_i	control element
$\underline{u}$	control vector
$\underline{u}_1$	control vector defined in (3-10)
$\underline{u}_2$	control vector defined in (3-10)
$\underline{v}$	left eigenvector
$\underline{\underline{V}}$	left eigenvector matrix

$\underline{w}$	right eigenvector
$\underline{\underline{W}}$	right eigenvector matrix
W_1	first effect holdup
W_2	second effect holdup
x_i	state variable
$\underline{x}$	state vector
$\underline{y}$	modal state vector
$\underline{z}$	integrated state vector

<u>Greek</u>

α_i	element of vector $\underline{\alpha}$
$\underline{\alpha}$	column vector defined in (2-17)
δ_i	element of vector $\underline{\delta}$
$\underline{\delta}$	column vector defined in (2-17)
$\underline{\underline{\Delta}}$	discrete control coefficient matrix
$\underline{\underline{\theta}}$	discrete disturbance coefficient matrix
$\underline{\underline{\phi}}$	discrete state coefficient matrix
λ	open-loop eigenvalue
Λ	set of eigenvalues
$\underline{\underline{\Lambda}}$	eigenvalue matrix
Π	product
ρ	desired eigenvalue
$\sum$	sum
τ	time

Superscript

•	time derivative
'	perturbation variable
^	augmented vector or matrix
i	recursive step counter
T	vector or matrix tanspose
-1	matrix inversion

Subscript

–	vector
=	matrix
i	element counter or run counter
j	element counter
p	size of matrix partition

Symbol

< >	dot product
{ }	set

Abbreviation

det	determinant
diag	diagonal
P	proportional control
PI	proportional plus integral control
sign	signum

ss	steady state
+20% CF	20% step increase in feed concentration
-20% CF	20% step decrease in feed concentration
+20F	20% step increase in feed flow rate
+20% TF	20% step increase in feed temperature

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APPENDIX

DERIVATION OF MATRICES $\hat{\underline{\underline{\phi}}}(T)$, $\hat{\underline{\underline{\Delta}}}(T)$ and $\hat{\underline{\underline{\theta}}}(T)$ FROM THE CONTINUOUS AUGMENTED SYSTEM

If one directly discretizes the continouous augmented system given by Equation (3-2) following standard procedures in the literature [25], the coefficient matrices, $\hat{\underline{\underline{\phi}}}(T)$, $\hat{\underline{\underline{\Delta}}}(T)$ and $\hat{\underline{\underline{\theta}}}(T)$ are expressed by:

$$\hat{\underline{\underline{\phi}}}(T) = e^{\hat{\underline{\underline{A}}}T} \tag{A-1}$$

$$\hat{\underline{\underline{\Delta}}}(T) = \left(\int_{o}^{T} e^{\hat{\underline{\underline{A}}}\tau} d\tau\right)\hat{\underline{\underline{B}}} \tag{A-2}$$

$$\hat{\underline{\underline{\theta}}}(T) = \left(\int_{o}^{T} e^{\hat{\underline{\underline{A}}}\tau} d\tau\right)\hat{\underline{\underline{D}}} \tag{A-3}$$

where matrices, $\hat{\underline{\underline{A}}}$, $\hat{\underline{\underline{B}}}$ and $\hat{\underline{\underline{D}}}$ are defined by Equations (3-4), (3-5) and (3-6) respectively. If the matrix $\underline{\underline{A}}$ in Equation (3-4) is non-singular, the right and left eigenvector matrices, $\hat{\underline{\underline{W}}}$ and $\hat{\underline{\underline{V}}}$ of matrix $\hat{\underline{\underline{A}}}$ can be related to the corresponding matrices $\underline{\underline{W}}$ and $\underline{\underline{V}}$ of matrix $\underline{\underline{A}}$ by [32];

$$\hat{\underline{\underline{W}}} = \left[\begin{array}{c|c} \underline{\underline{W}} & \underline{\underline{0}} \\ \hline \underline{\underline{T}}_r \underline{\underline{W}}\underline{\underline{\Lambda}}^{-1} & \underline{\underline{I}}_r \end{array}\right] \tag{A-4}$$

$$\hat{\underline{\underline{V}}}^T = \left[\begin{array}{c|c} \underline{\underline{V}}^T & \underline{\underline{0}} \\ \hline -\underline{\underline{T}}_r \underline{\underline{A}}^{-1} & \underline{\underline{I}}_r \end{array}\right] \tag{A-5}$$

where $\underline{\underline{\Lambda}}$ is a diagonal matrix whose diagonal elements are the eigenvalues of matrix $\underline{\underline{A}}$. Furthermore, the matrix exponential, $e^{\hat{\underline{\underline{A}}}\tau}$, can

be expressed by the following relation,

$$e^{\hat{\underline{\underline{A}}}\tau} = \hat{\underline{\underline{W}}}\, e^{\underline{\underline{\Lambda}}\tau}\, \hat{\underline{\underline{V}}}^T \tag{A-6}$$

with

$$\hat{\underline{\underline{\Lambda}}} = \left[\begin{array}{c|c} \underline{\underline{\Lambda}} & \underline{\underline{0}} \\ \hline \underline{\underline{0}} & \underline{\underline{0}} \end{array}\right] . \tag{A-7}$$

Substituting Equations (A-4), (A-5) and (A-7) into Equation (A-5) and rearranging the resulting expression gives,

$$e^{\hat{\underline{\underline{A}}}\tau} = \left[\begin{array}{c|c} e^{\underline{\underline{A}}\tau} & \underline{\underline{0}} \\ \hline \underline{\underline{T}}_r \underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}\tau} - \underline{\underline{I}}_n) & \underline{\underline{I}}_r \end{array}\right] \tag{A-8}$$

Hence, the integral expression in Equation (A-2) can be readily evaluated and given by,

$$\int_o^T e^{\underline{\underline{A}}\tau}\, d\tau = \left[\begin{array}{c|c} \underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}T} - \underline{\underline{I}}_n) & \underline{\underline{0}} \\ \hline \underline{\underline{T}}_r \underline{\underline{A}}^{-1}\{\underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}T} - \underline{\underline{I}}_n) - T\underline{\underline{I}}_n\} & T\underline{\underline{I}}_r \end{array}\right] \tag{A-9}$$

From Equations (A-1) and (A-8), matrix $\hat{\underline{\phi}}(T)$ is given by

$$\hat{\underline{\phi}}(T) = \left|\begin{array}{c|c} e^{\underline{\underline{A}}T} & \underline{\underline{0}} \\ \hline \underline{\underline{T}}_r \underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}T} - \underline{\underline{I}}_n) & \underline{\underline{I}}_r \end{array}\right| \tag{A-10}$$

After substituting Equation (A-9) into Equations (A-2) and (A-3) and using Equations (3-5) and (3-6), matrices $\hat{\underline{\underline{\Delta}}}(T)$ and $\hat{\underline{\underline{\theta}}}(T)$ are expressed by:

$$\hat{\underline{\underline{\Delta}}}(T) = \left[\begin{array}{c} \underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}T} - \underline{\underline{I}}_n)\underline{\underline{B}} \\ \hdashline \underline{\underline{T}}_r\underline{\underline{A}}^{-1}\{\underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}T} - \underline{\underline{I}}_n)\underline{\underline{B}} - T\underline{\underline{B}}\} \end{array}\right] \quad \text{(A-11)}$$

$$\hat{\underline{\underline{\theta}}}(T) = \left[\begin{array}{c} \underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}T} - \underline{\underline{I}}_n)\underline{\underline{D}} \\ \hdashline \underline{\underline{T}}_r\underline{\underline{A}}^{-1}\{\underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}T} - \underline{\underline{I}}_n)\underline{\underline{D}} - T\underline{\underline{D}}\} \end{array}\right] \quad \text{(A-12)}$$

From the equations given below Equation (2-33), it can be readily seen that:

$$\underline{\underline{\phi}}(T) = e^{\underline{\underline{A}}T} \quad \text{(A-13)}$$

$$\underline{\underline{\Delta}}(T) = \underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}T} - \underline{\underline{I}}_n)\underline{\underline{B}} \quad \text{(A-14)}$$

$$\underline{\underline{\theta}}(T) = \underline{\underline{A}}^{-1}(e^{\underline{\underline{A}}T} - \underline{\underline{I}}_n)\underline{\underline{D}} \quad . \quad \text{(A-15)}$$

Substitution of Equations (A-13) - (A-15) into Equations (A-10) - (A-12) respectively gives the expressions given by Equations (3-37) - (3-39).